Computational Methods For Handling Systems of Simultaneous Equations

WITH APPLICATIONS TO AGRICULTURE

by Joan Friedman and
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PREFACE

The basic theory underlying the use of systems of simultaneous equations in economic research was developed about 1943. Much of the early literature on this subject assumed a knowledge of higher mathematics and most of the applications of the method were made by research workers who had such knowledge. More recently, descriptions of this general approach that presume only a limited knowledge of mathematics have been published; and the method itself is beginning to be used by research workers who have little or no knowledge of calculus and matrix algebra.

Descriptions of the computations involved in handling systems of simultaneous equations are given in several books. Most of these, however, assume some knowledge of matrix manipulation, and all that have come to the attention of the authors omit many steps and fail to provide adequate coverage of the many special situations that are likely to arise.

This handbook is designed to provide a complete description of the steps involved in the more common types of problems and to illustrate them in a way that will be clear to research and clerical workers who have an acquaintance only with standard methods for handling single equation multiple regression analyses. Some knowledge of determinants and matrices is required for a number of problems, but these aspects are discussed in considerable detail before they are applied.

As some statistical computing units may prefer to use a comparable approach for all problems of this sort, whether they involve a single or a simultaneous set of equations, a method of handling ordinary least squares multiple regression analyses is given which utilizes the same initial steps as those for systems of simultaneous equations. This method is believed to be more efficient than those now commonly in use; it is easier for beginners to understand than those based essentially on the Doolittle approach, as no back solution is required. The description of this method should be clear to any clerical worker who is acquainted with the obtaining of sums of squares and cross products.

The general approach used in this handbook for systems of equations is given in Chapter 4 of _A Textbook of Econometrics_ by Lawrence R. Klein, and in Chapter 10 by Chernoff and Divinsky of _Studies in Econometric Method_, Cowles Commission for Research in Economics Monograph 14. Minor modifications have been made in methods that these authors suggest. These modifications and sources for other material are indicated by footnote. Suggestions offered by Frederick V. Waugh and Glenn L. Burrows, both of the Agricultural Marketing Service, and by Clifford Hildreth of Michigan State University, were particularly helpful.

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CONTENTS

COMPUTATIONAL METHODS FOR HANDLING SYSTEMS OF SIMULTANEOUS EQUATIONS
With Applications to Agriculture

by

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One of the purposes of this handbook is to provide a standard method of approach for handling any problem that involves the estimation of structural coefficients for economic relationships whether they are derived from a single equation or a system of simultaneous equations. In connection with the single equation approach, an example involving 5 variables is shown. The computations required for other numbers of variables are obvious.

When working with systems of simultaneous equations, modifications are required depending on (1) whether particular equations are just identified or overidentified and (2) the number of endogenous and predetermined variables in each. Methods for determining the degree of identification are given on page 28; table 11 on page 51 shows the exact steps to be used for any given situation. Examples are worked out in detail at each point that it appeared confusion might arise. In the past, many analysts have used the method of reduced forms to handle systems of equations for which each equation is just identified. We suggest, instead, use of a method that represents a slight modification of that used for equations that are overidentified. The structural coefficients obtained by either method are identical.

In addition to obtaining estimates of the various coefficients as such, most analysts want some indication of the probable sampling errors in these coefficients and in forecasts made from the analysis. If the independent variables used and the unexplained residuals from the analysis meet certain rather rigid specifications, such measures are available for single equation analyses regardless of the number of observations used in the study, and methods for obtaining them are given here. Exact estimates for such measures when working with systems of simultaneous equations, or for single equation analyses when based on the kind of data usually used in economic research, have been developed only for the case of an infinitely large sample.

Alternative methods for just identified equations which presumably would give identical answers with respect to the standard errors of the coefficients if applied to an infinitely large sample yield quite different answers when applied to analyses based on samples of the size commonly encountered in studies relating to economic data. We have outlined one of these methods for obtaining standard errors of the coefficients for equations that are just identified. Based on a study of a limited number of empirical examples, estimates obtained by this method range from 0.25 to 0.9 times as large as those given by an equally good method as applied to infinitely large samples. Conventional t-tests cannot be carried out with such estimates but, in a rather general way, they should give some indication of the probable magnitude of sampling fluctuations in the coefficients.

In contrast to these results, Wagner (21) 1/ used a Monte Carlo approach to study the sampling variability of coefficients that relate to equations that are overidentified, based on 100 samples of 20 observations each. These results suggest that the t-distribution may apply to such estimates and their computed standard errors. Further research will be required to ascertain whether this holds in general or only for the particular model that he studied.

In view of these problems with respect to standard errors of structural coefficients, we do not give a formula for the standard error of forecast from a system of simultaneous equations.

As many computations are required for complex systems of equations, it frequently is convenient to adjust the sums of squares and cross products in such a way as to make the sums of squares nearly equal to 1. This is desirable for any problem, although less important for simple than for complex ones. Methods of making this adjustment are discussed on page 6. All examples are based on the assumption that such an adjustment is used.

In general, the carrying of 9 decimals, particularly for problems involving many variables, is recommended to avoid the necessity of using a "floating" decimal point. In all cases, sufficient decimals should be carried for a minimum of 4 significant figures to appear in any computation. Some calculating machines do not provide full carryover for this number of decimals unless they are equipped with a special attachment. Errors caused by not having full carryover are important chiefly in connection with some types of negative multiplication. Clerical workers who plan to use 9 decimals in their computations should ascertain whether their machines are equipped to provide full carryover and, if not, should consult their manufacturer's representative. To save space in the tables included in this handbook, fewer decimals are shown than were actually used in the computations. Because of this, some computations appear to be slightly in error.

This handbook is designed to show how to make the necessary computations when working with alternative types of analyses. Material relating to interpretation has in general been omitted. The reader is referred to standard texts on statistics and econometrics. Specific references are given on certain topics.

A 5-VARIABLE MULTIPLE REGRESSION PROBLEM

This example is taken from a study by Lowenstein and Simon (15); it deals with factors that affect the domestic mill consumption of cotton. Logarithms of the following variables for the years 1921-40 and 1948-52 were used in the computations shown here:

1/ Underlined numbers in parentheses refer to Literature Cited, p. 87.

X_1 - Domestic mill consumption of cotton per capita, pounds

X_2 - Deflated disposable income per capita divided by 10, dollars

X_3 - Change in deflated disposable income per capita from the preceding year, dollars

X_4 - Mill consumption of synthetic fibers per 100 persons, pounds

X_5 - Deflated price per pound of Middling 7/8 inch cotton at the 10 spot markets, year beginning the preceding July, cents.

Since the regression equation was based on logarithms of the variables, coding of X_2 and X_4 affected only the constant term. The decoded value is given in their article whereas all of the coefficients shown in this section of the handbook apply to the coded variables expressed in logarithms.

Obtaining the Augmented Sums of Squares and Cross Products

The first step in the solution of any problem of this sort is to compute the "augmented" sums of squares and cross products or moments. Use of augmented moments is suggested to avoid rounding errors involved in obtaining arithmetic means. As used in this connection, an augmented moment equals the actual moment multiplied by the number of observations in the sample, here designated as N. In working with augmented moments, the sums of squares and cross products in terms of original values are cumulated directly on the calculating machine as for any problem of this type. The total for the observations included in the analysis is then multiplied by N, or the number of such observations. The correction factor for an augmented sum of squares equals the square of the sum of the series. The correction factor for an augmented cross product equals the product of the sum for each series. Subtraction of the correction factor from the augmented sum gives the augmented sum in terms of deviations from the respective means. These computations for the 5-variable regression problem are illustrated in table 1. It should be noted that X_1, the dependent variable, is written first.

A check column should always be carried in these and other computations. This is obtained by computing a "new" variable, Σ, for each year or observation in the analysis; this variable equals the sum of all of the variables for that observation. To check the computations involved in obtaining Σ, the sum for each of the variables, including the variable Σ, over all of the years included in the analysis is obtained. These sums constitute the first row of table 1. The sum of these sums for the variables other than Σ should exactly equal the sum of Σ. If they do, the computations involved in obtaining Σ are correct. The second row in table 1--the means--is obtained by dividing the sums in the first row by N, the sample size, which, for this example, equals 25. Cross-products for Σ with the other variables in the analysis are obtained in the usual way and are shown in the last column. The check for each row is carried out by computing the sum of all the items in the row, except for the item in the last, or Σ, column. For example,

Table 1.--Computation of augmented sums of squares and cross products 1/

Item	X_1 (1)	X_2 (2)	X_3 (3)	X_4 (4)	X_5 (5)	Σ (6)
Sum	35.1628	48.5788	0.1299	54.7388	30.9489	169.5594
Mean 2/	1.4065	1.9431	.0051	2.1895	1.2379	6.7823
1. Extensions with X_1:						
$\Sigma X_i X_1$	49.5412	68.4178	.2060	77.2481	43.5868	239.0001
$N\Sigma X_i X_1$	1,238.5320	1,710.4465	5.1500	1,931.2038	1,089.6718	5,975.0044
$\Sigma X_i \Sigma X_1$	1,236.4232	1,708.1681	4.5701	1,924.7731	1,088.2527	5,962.1874
Difference	2.1088	2.2784	.5799	6.4306	1.4190	12.8169
2. Extensions with X_2:						
$\Sigma X_i X_2$		94.6044	.2690	107.2316	60.2614	330.7844
$N\Sigma X_i X_2$		2,365.1117	6.7259	2,680.7915	1,506.5362	8,269.6121
$\Sigma X_i \Sigma X_2$		2,359.9027	6.3137	2,659.1512	1,503.4647	8,237.0006
Difference		5.2090	.4121	21.6403	3.0714	32.6114
3. Extensions with X_3:						
$\Sigma X_i X_3$			.0276	.3222	.2035	1.0284
$N\Sigma X_i X_3$			.6904	8.0553	5.0883	25.7101
$\Sigma X_i \Sigma X_3$			.0168	7.1144	4.0224	22.0376
Difference			.6735	.9408	1.0659	3.6725
. Extensions with X_4:						
$\Sigma X_i X_4$				126.3011	67.6013	378.7045
$N\Sigma X_i X_4$				3,157.5288	1,690.0334	9,467.6129
$\Sigma X_i \Sigma X_4$				2,996.3460	1,694.1122	9,281.4971
Difference				161.1827	-4.0788	186.1158
. Extensions with X_5:						
$\Sigma X_i X_5$					38.7390	210.3922
$N\Sigma X_i X_5$					968.4752	5,259.8050
$\Sigma X_i \Sigma X_5$					957.8387	5,247.6909
Difference					10.6365	12.1141

1/ The computations were performed with 9 decimal places, of which only 4 appear in the table; therefore, some of the computations may appear to be slightly in error.

2/ For this example, N = 25.

in the second row, the check is obtained by adding the means for X_1, X_2, X_3, X_4, and X_5. This should equal, except for rounding errors, the item in the Σ column, and if true, this is indicated by a √ placed next to that item. 2/

The terms in the lower left-hand part of the table are omitted. But in order to check the computations in all sections after the first, these omitted terms must be included. For example, the computation of the check for the first row of the third section of table 1 is given by:

$$0.2060 + 0.2690 + 0.0276 + 0.3222 + 0.2035 = 1.0284$$

The terms omitted from this row in the table, 0.2060 and 0.2690, are obtained, respectively, from the first row of the first section and the first row of the second section of column (3), the column in which the first written term of the row, 0.0276, appears. In general, for the ith row of any section, the omitted terms to the left of any given term, call it m, are obtained from the ith row of each section of the column in which m appears.

If a discrepancy due to a rounding error should occur, the sum across the row is considered as the correct figure and the figure originally shown in the Σ column is corrected accordingly. This corrected value is used in further computations. The following tabulation, showing the original computations with nine decimals for the items in the lower right corner of table 1, illustrates this point:

Σ column for section 5 of table 1

With 4 decimals	With 9 decimals
210.3922 √	210.39220317~~7~~6
5,259.8050 √	5,259.805079400 √
5,247.6909 √	5,247.690950246 √
12.1141 √	12.114129154 √

In the calculation with nine decimals, Σ in the first row equals 210.392203177. The result obtained by adding across the row is 210.392203176. Therefore the ninth decimal place is changed from a 7 to a 6. The corrected value, 210.392203176, is multiplied by N to give 5,259.805079400 in the second row. The above example illustrates a further point; since only a limited number of decimals are shown in this handbook, a √ was placed after <u>all</u> items in the Σ column that serve as checks. However, rounding errors do occur in some of these items. These result in part because the omitted figures were dropped without rounding.

2/ Rounding errors are usually taken to mean a discrepancy in the final decimal place. In some computations, the number of significant figures in the items operated upon is a further consideration.

If an error is made in computing the sums of squares and cross products, the following method is more efficient than a direct recomputation as a means of locating the error. Suppose that the checking operation indicates that an error has been made in obtaining the extensions with X_1. Continue to calculate the extensions with X_2 and, if the check for this indicates that no error has been made, we know that the augmented moment between X_1 and X_2 in the first section is correct. Similarly, if the extensions with X_3 check, we know that the augmented moment between X_1 and X_3 in the first section is correct. If all other extensions check, the mistake is in the computation of the sum of the squares for X_1. If one of the extensions does not check, recomputation of the corresponding element in the first section is indicated. A similar procedure is used if the initial error occurs in an extension other than with X_1.

Adjustments to Make the Sums of Squares Nearly Equal 1

It is a great convenience in computations to have all the elements on the main diagonal close to 1. In making this adjustment we are concerned only with the last row in each section of table 1. A set of values that are powers of 10, the k_i, where i is the variable to which it applies, is chosen such that when the sum of squares for the variable is multiplied by the square of the k_i the answer lies between 0.1 and 10. The value $(k_i)^2$ is referred to as the adjustment factor. The k_i are shown in the second column of table 2. They are determined in the following manner: In table 1, note that the sums of squares for X_1, X_2, and X_3, respectively, lie between 0.1 and 10; therefore the adjustment factor equals $(1.0)^2$ or 1.0 and k equals 1.0 for X_1, X_2, and X_3. For X_4, however, the sum of the squares equals 161.1827 and it must be multiplied by an adjustment factor of $(0.1)^2$ or 0.01 to bring it between 0.1 and 10; therefore k_4 equals 0.1. If the sum of squares for X_4, for example, had been 1611.827, the adjustment factor would be $(0.01)^2$ or 0.0001 and k_4 would equal 0.01. The adjustment factors for the cross products, the k_ik_j, are obtained by multiplying the k's for the variables involved. For example, k_3k_4, the adjustment factor for Σx_3x_4, equals (1.0) (0.1) = 0.1. The k_ik_j are shown in the right-hand section of table 2.

Table 2.- Adjustment factors

Variable	Value of k_i	k_ik_j for -				
		X_1	X_2	X_3	X_4	X_5
X_1	1.0	1.00	1.00	1.00	0.10	0.10
X_2	1.0		1.00	1.00	.10	.10
X_3	1.0			1.00	.10	.10
X_4	.1				.01	.01
X_5	.1					.01

Adjusted augmented moments are obtained by multiplying the augmented moment, the terms in the last row of each section of table 1, by its appropriate adjustment factor from table 2.

The adjustment process is important, naturally, only when the k_i differ considerably from one. Particularly when working with logarithms or first differences of logarithms, all of the k_i normally are close to one. Some computing units may prefer to adopt a general rule that adjustments are made only when at least one of the k_i lies outside the range 0.1 to 10. Had this rule been followed, adjustments would not have been made for this analysis.

The steps involved in obtaining the adjusted augmented moments are exactly the same for single and multiple equation analyses.

Obtaining Multiple and Partial Regression and Correlation Measures

The method of determining multiple regression constants discussed in the following pages differs in these two important ways from that given in some of the standard statistical textbooks: (1) The use of D, the inverse of the complete moment matrix, 3/ and (2) the computation of the inverse using a variation of the Doolittle method that omits the conventional back solution. 4/

Steps involved in the forward solution of the Doolittle method are given here in full detail as an aid to readers who are unacquainted with this method. Experience with our central computing unit demonstrated this as the easiest way to learn how to carry out these operations. Once the general approach is learned, many of the computations shown individually in table 3 can be cumulated directly in the calculating machine. Use is made of all possible shortcuts of this kind in the so-called abbreviated Doolittle method. This is the method described by Klein (13, pp. 151-155). An example based on it is shown in the appendix of this handbook. The so-called Crout method makes use of similar shortcuts and is an equally efficient method for solving systems of simultaneous equations or inverting matrices. This method is described in detail in the appendix, p. 95.

Computations involved in the forward solution of the Doolittle method are shown in table 3 and are as follows:

In rows (1) - (5), columns (1) - (5), enter the adjusted augmented moments computed above. The reader will note that the X's are listed in numerical order; in the method used by Ezekiel, X_1 is placed after the last independent

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3/ This approach, suggested to the authors by Frederick V. Waugh, Director, Agricultural Economics Division, Agricultural Marketing Service, substantially reduces the number of calculations necessary for the estimation of the various multiple regression coefficients, particularly the partial correlation coefficients. The method explained by Ezekiel (8), for example, is based upon the computation of the inverse of a matrix using only the moments for the independent variables. Ezekiel refers to the inverse of this as the C matrix, the elements of which are the c_{ij}.

4/ This approach was suggested to the authors by Daniel B. Suits of the Department of Economics, University of Michigan.

variable. Computations involved in obtaining table 2 and the adjusted augmented moments should be carefully checked as no automatic checks are available for these steps.

Additional columns, I_i, one for each variable in the analysis, are added in columns (7) - (11). The makeup of these is obvious from the table.

As an alternative, data shown in the upper section of table 3 can be recorded directly as the first row of each subsequent section.

In this forward solution we carry two check columns: Σ_x, column (6), for that part of the solution concerning the x's; and Σ_I, column (12), for that part of the solution concerning the I's. For the upper section of table 3, that is, rows (1) - (5), these columns are obtained in the following way: The element in the ith row of the Σ_x column is obtained by adding together the elements in the ith row of columns (1) - (5), including the omitted elements. The element omitted in the ith row and jth column can be found in the jth row of the ith column: For example, the omitted element in row (4), column (3), is the element in row (3), column (4), namely, 0.0940. The element in row (4) of the Σ_x column is given by: 0.6430 + 2.1640 + 0.0940 + 1.6118 + (-0.0407) = 4.4722. The element in the ith row of the Σ_I column is obtained by adding the elements in the ith row of columns (7) - (12). Because of the makeup of the columns, however, each element in these rows of the Σ_I column equals 1. In the computations outlined below, Σ_x and Σ_I are treated as additional variables, with all the operations performed upon them.

Only the second row in the first section and the last two rows in each succeeding section of the solution are checked. This is done in two parts, one for the x's and one for the I's. In order to check the computations in either of these rows in the x part of the forward solution, sum all the elements <u>in that row</u> for the x columns and compare that sum with the element in the Σ_x column for that row. There is no question of omitted elements here. These figures should be identical, except for rounding errors. If they are identical, this is indicated by a √. Where a discrepancy occurs due to a rounding error, the sum across the row replaces the element in the Σ column and is used in further computations. (See p. 5.) The check on the computations in the I section is obtained in like manner; that is, sum the elements in the ith row, columns (7) to (11), and compare that sum with the element in the ith row of the Σ_I column.

We now consider computations involved in each row of the lower sections of the forward solution in table 3.

<u>Row (1)</u>.--Copy row (1) from the upper section of table 3.

<u>Row (1")</u>.--Divide row (1) by its first term, that is by 2.1088, and perform the check. For computational purposes, it is more efficient to compute 1/2.1088 = 0.4741, lock it in the calculating machine, and multiply each item of row (1) by it.

<u>Row (2)</u>.--Copy row (2) of the upper section of table 3.

Table 3.--Obtaining partial and multiple regression and correlation measures for a 5 - variable multiple regression problem 1/

Forward solution

Row	x_1 (1)	x_2 (2)	x_3 (3)	x_4 (4)	x_5 (5)	Σ_x (6)	I_1 (7)	I_2 (8)	I_3 (9)	I_4 (10)	I_5 (11)	Σ_I (12)
Moments with--												
(1) x_1	2.1088	2.2784	0.5799	0.6430	0.1419	5.7521	1	0	0	0	0	1
(2) x_2		5.2090	.4121	2.1640	.3071	10.3708	0	1	0	0	0	1
(3) x_3			.6735	.0940	.1065	1.8663	0	0	1	0	0	1
(4) x_4				1.6118	-.0407	4.4722	0	0	0	1	0	1
(5) x_5					.1063	.6212	0	0	0	0	1	1
(1)	2.1088	2.2784	0.5799	0.6430	0.1419	5.7521	1	0	0	0	0	1
(1")	1.	1.0804	.2750	.3049	.0672	2.7276 ✓	.4741	0	0	0	0	.4741 ✓
(2)		5.2090	.4121	2.1640	.3071	10.3708	0	1	0	0	0	1
(1)(-1.0804)		-2.4615	-.6265	-.6947	-.1533	-6.2146	-1.0804	0	0	0	0	-1.0804
(2')		2.7474	-.2144	1.4692	.1538	4.1561 ✓	-1.0804	1	0	0	0	- .0804 ✓
(2")		1.	-.0780	.5347	.0559	1.5127 ✓	- .3932	.3639	0	0	0	- .0292 ✓
(3)			.6735	.0940	.1065	1.8663	0	0	1	0	0	1
(1)(-0.2750)			-.1595	-.1768	-.0390	-1.5819	- .2750	0	0	0	0	- .2750
(2')(0.0780)			-.0167	.1146	.0120	.3243	- .0843	.0780	0	0	0	- .0062
(3')			.4973	.0319	.0795	.6088 ✓	- .3593	.0780	1	0	0	.7187 ✓
(3")			1.	.0641	.1599	1.2241 ✓	- .7225	.1569	2.0106	0	0	1.4450 ✓
(4)				1.6118	-.0407	4.4722	0	0	0	1	0	1
(1) (-0.3049)				-.1960	-.0432	-1.7540	- .3049	0	0	0	0	- .3049
(2')(-0.5347)				-.7857	-.0822	-2.2225	.5777	-.5347	0	0	0	.0429
(3')(-0.0641)				-.0020	-.0051	.0390	.0230	-.0050	-.0641	0	0	- .0461
(4')				.6279	-.1714	.4565 ✓	.2958	-.5397	-.0641	1	0	.6919 ✓
(4")				1.	-.2729	.7270 ✓	.4711	-.8595	-.1021	1.5924	0	1.1018 ✓
(5)					.1063	.6212	0	0	0	0	1	1
(1)(-0.0672)					-.0095	-.3870	- .0672	0	0	0	0	- .0672
(2')(-0.0559)					-.0086	-.2326	.0604	-.0559	0	0	0	.0045
(3')(-0.1599)					-.0127	-.0974	.0574	-.0124	-.1599	0	0	- .1149
(4')(0.2729)					-.0467	.1246	.0807	-.1473	-.0175	.2729	0	.1838
(5')					.0286	.0286 ✓	.1314	-.2158	-.1775	.2729	1	1.0111 ✓
(5")					1.	1. ✓	4.5847	-7.5270	-6.1904	9.5205	34.8749	35.2627 ✓

Calculation of partial coefficients / Calculation of D matrix

								(1)	(2)	(3)	(4)	(5)	
(6) $b_{1j.} = -d_{1j}/d_{11}$	-1.	0.8909	0.8241	-0.9063	-2.4119	-2.6032 ✓	(1)	1.9008	-1.6934	-1.5665	1.7227	4.5847	4.9482 ✓
(7) $d_{11}d_{jj}$	3.6130	4.6850	5.9230	7.9671	66.2907	88.4790 ✓	(2)		2.4647	1.5481	-2.9143	-7.5270	-8.1220 ✓
(8) d^2_{1j}	3.6130	2.8678	2.4540	2.9679	21.0198	32.9228	(3)			3.1160	-1.7920	-6.1904	-4.8848 ✓
(9) (7)-(8)	0	1.8171	3.4689	4.9992	45.2709	55.5562 ✓	(4)				4.1914	9.5205	10.7282 ✓
(10) (9)x(0.0138)	0	.0251	.0480	.0691	.6264	.7688 ✓	(5)					34.8749	35.2627 ✓
(11) $s_{b_{1j.}} = \sqrt{(10)}$	-	.1585	.2191	.2630	.7915	-							
(12) $r^2_{1j.} = (8) \div (7)$	-	.6121	.4143	.3725	.3170	-						Sd_{jj} =	46.5480

Calculation of multiple coefficients

(13) $$R^2_{1.2345} = \frac{d_{11}m_{11} - 1}{d_{11}m_{11}} = \frac{(1.9008)\ (2.1088) - 1}{(1.9008)\ (2.1088)} = 0.7505$$

$$s^2_{1.2345} = \frac{1}{NN'd_{11}} = \frac{1}{(25)(20)(1.9008)} = 0.0010 \qquad s_{1.2345} = 0.0324$$

Calculation of deadjusted coefficients

Row	Variable	b	s_b	k_i	$k_i' = \frac{k_i}{k_1}$	Deadj. b	Deadj. s_b	$\bar{X}_i$	Deadj. b $\bar{X}$
(14)	X_1	-	-	1.0	-	-	-	1.4065	-
(15)	X_2	0.8909	0.1585	1.0	1.0	0.8909	0.1585	1.9431	1.7312
(16)	X_3	.8241	.2191	1.0	1.0	.8241	.2191	.0051	.0042
(17)	X_4	-.9063	.2630	.1	.1	-.0906	.0263	2.1895	-.1984
(18)	X_5	-2.4119	.7915	.1	.1	-.2411	.0791	1.2379	-.2985
									1.2384

$s_{1.2345} = (0.0324)/(1) = 0.0324$

$a = 1.4065 - 1.2384 = 0.1680$

(19) $$X_1 = 0.1680 + \underset{(0.1585)}{0.8909}X_2 + \underset{(0.2191)}{0.8241}X_3 - \underset{(0.0263)}{0.0906}X_4 - \underset{(0.0791)}{0.2411}X_5$$

1/ These calculations were performed with 9 decimal places, of which only 4 appear in the table; therefore, some of the computations may appear slightly in error.

Row (1)(-1.0804).--Multiply row (1) by -1.0804. This factor is the element of row (1"), column (2), with its sign changed. Note that no figures are inserted in this section of the table in columns to the left of column (2).

Row (2').--Add row (2) and the following line and perform the check.

Row (2").--Divide row (2') by its first term, that is, by 2.7474, and perform the check. Or, multiply row (2') by 1/2.7474 = 0.3639.

Row (3).--Copy row (3).

Row (1)(-0.2750).--Multiply row (1) by -0.2750. This factor is the element of row (1"), column (3), with its sign changed.

Row (2')(0.0780).--Multiply row (2') by 0.0780. This factor is the element of row (2"), column (3), with its sign changed.

Row (3').--Add row (3) and the two rows following it and perform the check.

Row (3").--Divide row (3') by its first term, that is, by 0.4973, and perform the check. Or multiply row (3') by 1/0.4973 = 2.0106.

Row (4).--Copy row (4).

Row (1)(-0.3049).--Multiply row (1) by -0.3049. This factor is the element of row (1"), column (4), with its sign changed.

Row (2')(-0.5347).--Multiply row (2') by -0.5347. This factor is the element of row (2"), column (4), with its sign changed.

Row (3')(-0.0641).--Multiply row (3') by -0.0641. This factor is the element of row (3"), column (4), with its sign changed.

Row (4').--Add row (4) and the three rows following it and perform the check.

Row (4").--Divide row (4') by its first term, that is, by 0.6279 and perform the check. Or, multiply row (4') by 1/0.6279 = 1.5924.

Row (5).--Copy row (5).

Row (1)(-0.0672).--Multiply row (1) by -0.0672. This is the element of row (1"), column (5), with its sign changed.

Row (2')(-0.0559).--Multiply row (2') by -0.0559. This is the element of row (2"), column (5), with its sign changed.

Row (3')(-0.1599).--Multiply row (3') by -0.1599. This is the element of row (3"), column (5), with its sign changed.

Row (4')(0.2729).--Multiply row (4') by 0.2729. This is the element of row (4"), column (5), with its sign changed.

Row (5').--Add row (5) and the four rows following it and perform the check.

Row (5").--Divide row (5') by its first term, that is, by 0.0286, and perform the check. Or, multiply row (5') by 1/0.0286 = 34.8749.

This completes the forward solution.

Unfortunately, the checks do not guarantee that the correct multiplicand has been used; they only prove that the multiplications were carried out correctly. As a final check, it is suggested that the multiplicands shown in the stub be examined to make sure that the correct value was used and that these then be used to recheck the computations in the Σ_I column (column 12 in table 3). Experience in our central computing unit has indicated that occasionally a statistical clerk is interrupted between the computations involved in the x and the I part of the table and that the wrong multiplicand is used in the latter set of computations. It seems unlikely, however, that a wrong multiplicand would be used in the x part of the table and the correct one in the I part. When the abbreviated Doolittle solution is used (see appendix), this final check is not needed, as the computations are carried out on a column-by-column basis rather than a row-by-row basis.

D Matrix.--The D matrix is shown in table 3 immediately following the I part of the forward solution. Its computation involves the terms in the last 2 rows of each section in the I part of the forward solution. The element in the ith row and jth column of the D matrix, d_{ij}, is obtained by the following formula:

$$d_{ij} = (1,I_i)(1", I_j) + (2', I_i)(2", I_j) + (3', I_i)(3", I_j) + (4', I_i)$$
$$(4", I_j) + (5', I_i)(5", I_j)$$

where the first term within the parentheses refers to the row and the second, to the column designation of the elements in the forward solution. Therefore:

$$d_{11} = (1)(0.4741) + (-1.0804)(-0.3932) + (-0.3593)(-0.7225)$$
$$+ (0.2958)(0.4711) + (0.1314)(4.5847) = 1.9008$$

$$\text{and } d_{12} = (1)(0) + (-1.0804)(0.3639) + (-0.3593)(0.1569)$$
$$+ (0.2958)(-0.8595) + (0.1314)(-7.5270)$$

These sums should be cumulated directly in the calculating machine. A check column, Σ, is also carried in this computation. The elements in the Σ column, $d_i\Sigma$, are computed in the same way as any other element in the D matrix. In the general formula given above, I_j becomes Σ_I. That the sum across the ith row of D is identical (except for possible rounding errors growing out of the carrying of only 4 decimals) with the element in the ith row of the Σ column is indicated by a check mark. This checks the computation of the ith row. It will be noted that the elements in the lower part of the

D matrix have been omitted. These need not be computed, since $d_{ij} = d_{ji}$. In computing the check on the computations in rows after the first, however, these omitted elements must be included. For example, the check on the computation of the fourth row of the D matrix is given by:

$$1.7227 + (-2.9143) + (-1.7920) + 4.1914 + 9.5205 = 10.7282$$

The next to the last (fifth) column of the D matrix need not be computed, since it corresponds to the last row (5") in the I part of the forward solution.

All the usual measures of partial regression and correlation can be obtained easily from the D matrix. These calculations are shown in rows (6) - (12) of columns (1) - (5); column (6) is a check column.

<u>Partial Regression Coefficients</u>.--The calculation of the highest order partial regression coefficients, the "b's", is shown in row (6). This is done as follows:

Row (6): Divide each element of the first row of the D matrix, including the element in the Σ column, by the first element in the first row of D, and <u>change the sign</u> of the resulting quotient. Symbolically, $b_{1j}. = -d_{1j}/d_{11}$, where j refers to the subscript of the x's and the column of the D matrix. $b_{12.345}$, the coefficient of x_2, therefore equals $- d_{12}/d_{11}$ or $- (-1.6934)/1.9008 = 0.8909$. $b_{13.245}$, $b_{14.235}$, and $b_{15.234}$, the coefficients on x_3, x_4, and x_5, respectively, are obtained in like manner. That the sum across row (6) is identical (except for possible rounding error due to carrying only 4 decimals) with the element in the Σ column is indicated by a check mark. This checks the computation of the b's. Since x_1 is the dependent variable, no coefficient is attached to it. The -1 in row (6), column (1), and the figures in the following rows are written in order to check the computations.

<u>Standard Errors of the Regression Coefficients</u>.--The calculation of the standard errors of the highest order partial regression coefficients is shown in rows (7) - (11). This is done as follows:

Row (7): Compute $d_{11}d_{jj}$, that is, the product of the element in the first row and first column of D with the successive diagonal elements of D. For example, the element in the first or x_1 column is obtained by squaring d_{11}; the element in the second or x_2 column equals $d_{11}d_{22} = (1.9008)(2.4647) = 4.6850$; and the element in the Σ column is obtained by multiplying d_{11} by Sd_{jj}. Sd_{jj} is the sum of the diagonal elements of D and is shown in the last column of row (12). That the sum across row (7) is identical (except for possible rounding error) with the element in the Σ column is indicated by a check mark.

Row (8): Compute d_{1j}^2, that is, the square of each of the elements in the first row of the D matrix, excluding the element in the Σ column. The element in the Σ column of row (8) is the sum across the row. The check on this row is one of recomputation.

Row (9): Subtract each element of row (8) from the element in the corresponding column of row (7), including those in the Σ column. That the sum across row (9) is identical (except for possible rounding error) with the element in the Σ column is indicated by a check mark. This checks the computation of row (9).

Row (10): Compute $1/N'd_{11}^2$, where N' equals the sample size minus the total number of variables, and d_{11}^2 is the square of the element in the first row and first column of D. The value d_{11} is given in the first column of rows (7) and (8). In this example, N' equals 25-5 or 20 and d_{11}^2 equals 3.6130; therefore $1/N'd_{11}^2 = 1/72.2616 = 0.0138$. Multiply each element in row (9) by 0.0138, including that in the Σ column. That the sum across row (10) is identical (except for possible rounding error) with the item in the Σ column is indicated by a check mark.

Row (11): Compute the square root of the element in the corresponding column of row (10), except the element in the Σ column. The elements in row (11) are the standard errors of the coefficients in the corresponding column of row (6). The check is one of recomputation.

Coefficients of Partial Determination.--The calculation of the highest order coefficients of partial determination (the square of the partial correlation coefficient) is shown in row (12). This is done as follows:

Row (12): Divide each element in row (8) by the element in the corresponding column of row (7), except the Σ column. The elements in row (12) are the coefficients of partial determination. The element in row (12), column (2), for example, equals $r_{12.345}^2$. The check on this row is one of recomputation.

If the coefficients of partial correlation are desired, they can be obtained by taking the square root of the elements in row (12).

Coefficient of Multiple Determination.--$R_{1.2345}^2$, the coefficient of multiple determination, is obtained by the following formula:

$$R_{1.2345}^2 = \frac{d_{11}m_{11} - 1}{d_{11}\,m_{11}}$$

where d_{11} is the element in the first row and first column of the D matrix, and m_{11} is the adjusted augmented moment of x_1 on x_1, which is found in the first row and first column of the upper part of table 3. In this example, d_{11} equals 1.9008 and m_{11} equals 2.1088. Therefore, $R_{1.2345}^2 = \frac{(1.9008)(2.1088) - 1}{(1.9008)(2.1088)} = 0.7505$. The coefficient of multiple correlation, $R_{1.2345}$, if desired, can be obtained by taking the square root of the coefficient of multiple determination.

Standard Error of Estimate.-- $s_{1.2345}$, the standard error of estimate, is obtained by the following formula:

$$s_{1.2345} = \sqrt{\frac{1}{NN'd_{11}}}$$

where d_{11} is the element in the first row and first column of the D matrix, N is the sample size, and N' is N minus the total number of variables. In this example, d_{11} equals 1.9008, N equals 25, and N' equals 20. Therefore, $s^2_{1.2345} = \frac{1}{25(20)(1.9008)} = 0.0010$. $s_{1.2345}$, the standard error of estimate, equals the square root of this value or 0.0324. The N in this formula is required because of the use of augmented moments.

Regression Equation Based on Deadjusted Data.--Since the regression coefficients and their standard errors are computed on the basis of adjusted data, they must be deadjusted in order to apply to the original data. This deadjustment, carried out in rows (14) - (18), is as follows:

Column (1): Enter the variables in numerical order.

Column (2): Enter the regression coefficients, the b's, obtained in row (6). Note that no figure is entered for X_1.

Column (3): Enter the standard errors of the regression coefficients, obtained in row (11).

Column (4): Enter the appropriate values of k_i from table 2.

Column (5): Compute $k_i' = k_i/k_1$.

Column (6): The deadjusted b's are obtained by multiplying the b's, column (2), by their respective k_i'.

Column (7): The deadjusted standard errors of the b's are obtained by multiplying the s_b, column (3), by their respective k_i'.

Column (8): Enter the means of the variables from table 1.

Column (9): Computations in this column are used in obtaining the constant for the equation. Multiply the deadjusted b's, column (6), by the mean in the corresponding row of column (8) and add the figures in column (9), or cumulate the products directly in the machine. The constant in the equation, a, is obtained by subtracting the cumulated product from the mean of X_1, the element in row (14), column (8). Hence, a = 1.4065 - 1.2384 = 0.1680. This result can be recorded directly as the constant in the regression equation shown in row 19.

The final regression equation, in the following form, is shown in row (19):

$$X_1 = a + b_{12.345} X_2 + b_{13.245}X_3 + b_{14.235}X_4 + b_{15.234}X_5$$

The figures in the table within the parentheses are the standard errors of the respective regression coefficients.

The standard error of estimate, $s_{1.2345}$, also must be deadjusted. This is done by dividing $s_{1.2345}$ by k_1. The latter is given in row (14), column (4).

In our example, $k_1 = 1$; therefore, for this example, the standard error of estimate is the same on an adjusted or deadjusted basis. The indicated computation is shown at the end of row (14).

The coefficient of multiple determination need not be deadjusted.

The check in this section is one of recomputation.

If all of the k_i' equal one, columns (6) and (7) can be omitted. In this case, column (2) is used in place of column (6) in obtaining the constant in the equation in column (9).

Eliminating or Adding Variables

If one or more variables are to be eliminated or added, the measures of correlation and regression can be obtained without rerunning the analysis.

Eliminating Variables.--Application of the formula given below, which applies if one variable is to be eliminated, yields elements of a similar D matrix, $D)k($, for all variables except the omitted one, x_k. 5/ The elements of this matrix, the $d_{ij})k($, can be obtained by the formula:

$$d_{ij})k(= \frac{d_{ij}d_{kk} - d_{ik}d_{jk}}{d_{kk}}$$

where the d's are the elements of D. These $d_{ij})k($ values are used in place of the corresponding d_{ij} values in the computations beginning with row (6) of table 3. These computations are explained on p. 12.

For example, if x_4 were to be dropped from the previous analysis, we would compute the first row of $D)4($, that is, $d_{11})4($, $d_{12})4($, $d_{13})4($, and $d_{15})4($ by the formula:

$$d_{1j})4(= \frac{d_{1j}d_{44} - d_{14}d_{j4}}{d_{44}}$$

If we consider the adjusted augmented moments of x_1 with the other variables given in the first row of table 3 as m_{ij}, a check on the computation of the first row of $D)4($ is given by computing $m_{11}d_{11})4(+ m_{12}d_{12})4(+ m_{13}d_{13})4(+ m_{15}d_{15})4($. This sum should equal 1.

It is not necessary to compute the entire $D)4($ matrix. In addition to the first row, we need only compute the diagonal elements, that is, $d_{jj})4($, given by the formula:

$$d_{jj})4(= \frac{d_{jj}d_{44} - d_{j4}^2}{d_{44}}$$

5/ This formula was suggested by Frederick V. Waugh, Director, Division of Agricultural Economics, Agricultural Marketing Service.

The partial regression coefficients can be obtained by:

$$b_{1j.)4(} = - \frac{d_{1j)4(}}{d_{11)4(}}$$

Their standard errors are given by:

$$s_{b_{1j.)4(}} = \sqrt{\frac{d_{11)4(}\,d_{jj)4(} - d^{2}_{1j)4(}}{N'\,d^{2}_{11)4(}}}$$

The coefficients of partial determination equal:

$$r^{2}_{1j.)4(} = \frac{d^{2}_{1j)4(}}{d_{11)4(}\,d_{jj)4(}}$$

The coefficient of multiple determination equals:

$$R^{2}_{1.235} = \frac{d_{11)4(}\,m_{11} - 1}{d_{11)4(}\,m_{11}}$$

The standard error of estimate is given by:

$$s_{1.235} = \sqrt{\frac{1}{N\,N'\,d_{11)4(}}}$$

Similarly, if both x_r and x_k were to be eliminated, we would compute the elements of $D_{)kr(}$, the $d_{ij)kr(}$, as follows:

$$d_{ij)kr(} = \frac{d_{ij)k(}\,d_{rr)k(} - d_{ir)k(}\,d_{jr)k(}}{d_{rr)k(}}$$

Thus, if more than one variable is to be eliminated, the computations must be done in steps by eliminating them one at a time.

Use of the above formula is easy if only one variable is to be eliminated; it becomes more difficult as additional variables are dropped. Sometimes the analyst knows fairly well in advance which variables may need to be eliminated. If so, he should use them as the highest-numbered independent variables. If X_4 and X_5 were to be eliminated, this could be done by dropping columns (4), (5), (10) and (11) and rows (4) and (5) and their corresponding sections in the forward solution. The D matrix then could be easily recomputed and the remaining computations carried out as in any 3-variable analysis. New check sums for use in the computations beginning with row (6) probably would be advisable.

Adding Variables.--In general, it is easier to drop variables than to add them. Hence, as many variables as are likely to be used should be incorporated in the initial analysis; some of these can then be dropped if this appears advisable. At times, however, a variable will need to be added. Assume that the added variable is X_6. Use can be made of all of the computations already made in the forward solution. Columns are added between the former columns (5) and (6) and between columns (11) and (12), and a row (6) and a corresponding section are added in the forward solution. Figures in these columns can be filled in by performing the same sort of computations as were done previously. An additional product from the new section 6 will need to be added to each of the elements in the original D matrix, and a new column (6) and row (6) should be added. These steps can be checked by recomputation or by use of a new check sum. All of the coefficients should be recalculated, making use of the new D matrix and of new check sums.

Standard Errors of the Function and of Forecasts

The standard error of a point on the regression equation, or function, relates to a point on the regression surface corresponding to specified values of the independent variables. Its value differs from point to point, depending on the specific values assigned to these variables. This coefficient is used in two ways: (1) It is a convenient step in computing the standard error of a specific forecast, and (2) when the values assigned to the independent variables are all zero, it equals the standard error of the constant in the regression equation.

For a 5-variable multiple regression problem, the square of the standard error of a point on the regression equation, or function, is given by:

$$\begin{aligned} s^2_{F1.2345} = s^2_{1.2345} \Big[& \frac{1}{N} + Nc_{22}(X_2 - \bar{X}_2)^2 + Nc_{33}(X_3 - \bar{X}_3)^2 + Nc_{44}(X_4 - \bar{X}_4)^2 \\ & + Nc_{55}(X_5 - \bar{X}_5)^2 + 2Nc_{23}(X_2 - \bar{X}_2)(X_3 - \bar{X}_3) + 2Nc_{24}(X_2 - \bar{X}_2)(X_4 - \bar{X}_4) \\ & + 2Nc_{25}(X_2 - \bar{X}_2)(X_5 - \bar{X}_5) + 2Nc_{34}(X_3 - \bar{X}_3)(X_4 - \bar{X}_4) \\ & + 2Nc_{35}(X_3 - \bar{X}_3)(X_5 - \bar{X}_5) + 2Nc_{45}(X_4 - \bar{X}_4)(X_5 - \bar{X}_5) \Big] \end{aligned}$$

where $s^2_{1.2345}$ is the deadjusted value of the square of the standard error of estimate obtained by squaring the deadjusted standard error of estimate, N is the number of observations on which the analysis is based, 6/ and the c_{ij} are obtained from the elements of the D matrix shown in table 3 by the formula:

$$c_{ij} = \frac{d_{11}d_{ij} - d_{1i}d_{1j}}{d_{11}}$$

6/ N is required in the terms after the first within the brackets because of the use of augmented moments in the computations.

If these values of c_{ij} are substituted in the formula for the square of the standard error of the function, N and d_{11} appear in each of the products within the brackets. We can, therefore, rewrite the formula as:

$$s^2_{F_{1.2345}} = s^2_{1.2345}\left\{\frac{1}{N} + \frac{N}{d_{11}}\left[c''_{22}(X_2 - \bar{X}_2)^2 + c''_{33}(X_3 - \bar{X}_3)^2 + c''_{44}(X_4 - \bar{X}_4)^2 \right.\right.$$

$$+ c''_{55}(X_5 - \bar{X}_5)^2 + 2c''_{23}(X_2 - \bar{X}_2)(X_3 - \bar{X}_3) + 2c''_{24}(X_2 - \bar{X}_2)(X_4 - \bar{X}_4)$$

$$+ 2c''_{25}(X_2 - \bar{X}_2)(X_5 - \bar{X}_5) + 2c''_{34}(X_3 - \bar{X}_3)(X_4 - \bar{X}_4)$$

$$\left.\left. + 2c''_{35}(X_3 - \bar{X}_3)(X_5 - \bar{X}_5) + 2c''_{45}(X_4 - \bar{X}_4)(X_5 - \bar{X}_5)\right]\right\}$$

where $c''_{ij} = d_{11}d_{ij} - d_{1i}d_{1j}$. The computed c''_{ij} are shown in table 4.

Table 4.- c''_{ij} for the 5 variable multiple regression problem

Outline				Values			
c''_{22}	c''_{23}	c''_{24}	c''_{25}	1.8171	0.2898	-2.6222	-6.5433
	c''_{33}	c''_{34}	c''_{35}		3.4689	- .7076	-4.5845
		c''_{44}	c''_{45}			4.9992	10.1983
			c''_{55}				45.2709

c''_{22}, c''_{33}, c''_{44}, and c''_{55} were computed in row (9) of table 3, columns (2), (3), (4), and (5), respectively. The other c''_{ij} must be computed directly. For example, $c''_{23} = d_{11}d_{23} - d_{12}d_{13}$. Substituting the values from the D matrix, we obtain: c''_{ij} = (1.9008)(1.5481) - (-1.6934)(-1.5665) = 0.2898. The computation of the c''_{ij} can be checked by computing the following sums of products:

(a) $c''_{22}\, m_{22} + c''_{23}\, m_{23} + c''_{24}\, m_{24} + c''_{25}\, m_{25}$

(b) $c''_{23}\, m_{23} + c''_{33}\, m_{33} + c''_{34}\, m_{34} + c''_{35}\, m_{35}$

(c) $c''_{24}\, m_{24} + c''_{34}\, m_{34} + c''_{44}\, m_{44} + c''_{45}\, m_{45}$

(d) $c''_{25}\, m_{25} + c''_{35}\, m_{35} + c''_{45}\, m_{45} + c''_{55}\, m_{55}$

where m_{ij} is the adjusted augmented moment of x_i on x_j shown in rows (1) - (5), columns (1) - (5), of table 3. Each of these sums of products should equal d_{11}, except for possible rounding errors. For example, to check the first row of the

c''_{ij}, we compute (a): (1.8171)(5.2090) + (0.2898)(0.4121) + (-2.6222)(2.1640) + (-6.5433)(0.3071) = 1.9008. The second, third, and fourth rows are checked by computing (b), (c), and (d), respectively. These c''_{ij} are in adjusted terms.

For use in the formula for the standard error of a function, the c''_{ij} must be deadjusted. This is done by multiplying c''_{ij} by $k_i k_j$, the appropriate adjustment factor from table 2. For example, to deadjust c''_{24}, -2.6222, multiply by $k_2 k_4$, or 0.10. Therefore, the deadjusted value of c''_{24} = (-2.6222)(0.10) = -0.2622. The nature of the formula is such, however, that d_{11} is <u>never</u> deadjusted.

The means that are used in the formula are obtained from table (3), rows (15) - (18) of column (8). These are given on a deadjusted basis.

Inserting the deadjusted standard error of estimate, c''_{ij}, means, and the adjusted d_{11} in the formula for the square of the standard error of the function gives:

$$
\begin{aligned}
s^2_{F_{1.2345}} = 0.0010 \Bigg\{ \frac{1}{25} + \frac{25}{1.9008} \Big[& 1.8171\ (X_2 - 1.9431)^2 + 3.4689\ (X_3 - 0.0051)^2 \\
& + 0.0499(X_4 - 2.1895)^2 + 0.4527(X_5 - 1.2379)^2 \\
& + 2(0.2898)(X_2 - 1.9431)(X_3 - 0.0051) + 2(-0.2622)(X_2 - 1.9431)(X_4 - 2.1895) \\
& + 2(-0.6543)(X_2 - 1.9431\ (X_5 - 1.2379) + 2(-0.0707)(X_3 - 0.0051)(X_4 - 2.1895) \\
& + 2(-0.4584)(X_3 - 0.0051)(X_5 - 1.2379) + 2(0.1019)(X_4 - 2.1895)(X_5 - 1.2379) \Big] \Bigg\}
\end{aligned}
$$

The standard error of the function is obtained by inserting the specified values of X_2, X_3, X_4 and X_5 for any given observation and taking the square root of the result.

The standard error of a specific forecast is obtained from the following formula:

$$s_{x'_1} = \sqrt{s^2_{F_{1.2345}} + s^2_{1.2345}}$$

$$= \sqrt{s^2_{F_{1.2345}} + 0.0010}$$

where $s^2_{1.2345}$ is on a deadjusted basis.

Use of an Alternative Variable as the Dependent One

All measures of regression and correlation given in preceding sections are based on the use of X_1 as the dependent variable. If, after the analysis is run, it seems desirable to have one of the other variables, X_i, as the dependent one, the various statistical measures can be obtained from the original D matrix by use of the following:

The partial regression coefficients equal:

$$b_{ij.} = -\frac{d_{ij}}{d_{ii}}$$

If, for example, X_2 is to be used as the dependent variable in the 5-variable problem given above, we would compute: $b_{21.345}$, $b_{23.145}$, $b_{24.135}$, and $b_{25.134}$, where $b_{21.345} = -\frac{d_{21}}{d_{22}}$, etc.

The standard errors of the regression coefficients are given by:

$$s_{b_{ij.}} = \sqrt{\frac{d_{ii}d_{jj} - d_{ij}^2}{N' d_{ii}^2}}$$

For example, $$s_{b_{21.345}} = \sqrt{\frac{d_{22}d_{11} - d_{21}^2}{N' d_{22}^2}}$$

The coefficients of partial determination equal:

$$r_{ij.}^2 = \frac{d_{ij}^2}{d_{ii}d_{jj}}$$

For example, $$r_{21.}^2 = \frac{d_{21}^2}{d_{22}d_{11}}$$

The coefficient of multiple determination is given by:

$$R_{i.}^2 = \frac{d_{ii}m_{ii} - 1}{d_{ii}m_{ii}}$$

For example, $$R_{2.1345}^2 = \frac{d_{22}m_{22} - 1}{d_{22}m_{22}}$$

The standard error of estimate equals:

$$s_{i.} = \sqrt{\frac{1}{NN'd_{ii}}}$$

For example, $$s_{2.1345} = \sqrt{\frac{1}{NN'd_{22}}}$$

It should be noted that when variables are eliminated, added, or interchanged, the regression coefficients, their standard errors, and the standard error of estimate must be deadjusted before they can be applied to the original data. All of the formulas shown apply to adjusted values.

ELEMENTARY PRINCIPLES REGARDING MATRICES AND DETERMINANTS

In this section, we briefly discuss those elements of matrix algebra that are required for an understanding of the computational methods described later in this handbook. In some sections, discussion is fairly complete; in others, only a bare outline of essentials is given. The discussion is particularly incomplete in connection with determinants and the inverse of a matrix. More complete summaries of material that relates to matrices and determinants are given in Klein (13, pp. 324-341), Tintner (18, pp. 331-341), and elsewhere.

Definitions

A matrix is an array of numbers arranged in rows and columns. Certain rules of addition, subtraction, and multiplication that apply to matrices are described below.

In general terms, the matrix A can be written in the following way:

$$A = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} \cdots & a_{mn} \end{bmatrix}$$

where the subscripts attached to the a's indicate the row and column, respectively. For example, the first subscript on a_{32} means that the element is in the third row, and the second subscript means the element is in the second column of the A matrix. Since A has m rows and n columns, we call A an (m x n) matrix. When m=n, that is, when the number of rows is equal to the number of columns, we have a square matrix. A square matrix with n rows and n columns is called a matrix of order n.

Consider the (3 x 4) matrix A:

$$A = \begin{bmatrix} 5 & 3 & 2 & 1 \\ 4 & 2 & 1 & 3 \\ 6 & -1 & 0 & 2 \end{bmatrix}$$

Here $a_{13}=2$, $a_{23}=1$, $a_{33}=0$, etc.

Throughout this handbook, computations are performed with augmented moment matrices. An augmented moment matrix, $\bar{M}_{XX}$, is a matrix whose elements are the adjusted augmented sums of squares and cross products (or augmented moments) of the variables indicated in the subscripts, in this case, the X's. 7/ If, for

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7/ In future references to this type of matrix, the word "augmented" and the bar over the M are omitted.

example, we deal with 3 X's designated respectively X_1, X_2 and X_6 and the m_{ij} are the elements of M_{XX}, then m_{11} is the moment of x_1 on x_1, m_{12} is the moment of x_1 on x_2, m_{13} is the moment of x_1 on x_6, m_{21} is the moment of x_2 on x_1, m_{22} is the moment of x_2 on x_2, m_{23} is the moment of x_2 on x_6, m_{31} is the moment of x_6 on x_1, m_{32} is the moment of x_6 on x_2, and m_{33} is the moment of x_6 on x_6.

A <u>row vector</u> is a matrix that has one row and n columns.

Example: $B = (1\ -2\ 0)$ is a row vector with n=3 columns; here,

$b_{11} = 1$, $b_{12} = -2$, $b_{13} = 0$.

A <u>column vector</u> is a matrix that has m rows and one column.

Example:

$$C = \begin{bmatrix} 2 \\ 3 \\ -1 \\ 4 \end{bmatrix}$$ is a column vector with m=4 rows; here, $c_{11} = 2$, $c_{21} = 3$, $c_{31} = -1$, $c_{41} = 4$.

A <u>scalar</u> is a matrix that has one row and one column; that is, it is a matrix with one element, or an ordinary number.

Example: $D = 5$ is a scalar.

A <u>symmetrical matrix</u> is a square matrix in which all the corresponding elements above the main, or left to right, diagonal are equal to elements below the diagonal; that is, $a_{ij} = a_{ji}$.

Example:

$$E = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 4 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$ is a symmetrical matrix.

It should be noted that the augmented sums of squares and cross products used in multiple regression analysis form a symmetrical matrix.

The <u>unit or identity matrix</u>, <u>I</u>, is a square matrix in which all the elements along the main diagonal are 1, and all the nondiagonal elements are zero.

Example:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is a unit matrix of order 3.

The <u>transpose of the matrix A</u>, written A', is a matrix in which the rows of A are the columns of A', and the columns of A are the rows of A'.

Example: $$A' = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$ is the transpose of $$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 3 & 2 \end{bmatrix}$$

It should be noted that the transpose of a symmetrical matrix equals the matrix itself.

The transpose of a row vector is a column vector and vice versa. In this handbook, the term vector used alone always refers to a row vector; a column vector, therefore, is written as the transpose. This convention is commonly used by mathematicians.

Addition and Subtraction of Matrices

Addition and subtraction of matrices can be performed only if the matrices have the same number of rows and the same number of columns, respectively.

Addition of matrices is performed by adding their elements term by term.

Example:

$$A = \begin{bmatrix} 1 & 0 & 4 & 3 \\ 3 & 2 & -1 & -6 \\ 2 & 1 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 2 & 1 & 2 \\ 2 & 1 & 0 & 3 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

$$C = A + B = \begin{bmatrix} 1 & 2 & 5 & 5 \\ 5 & 3 & -1 & -3 \\ 3 & 2 & 2 & 3 \end{bmatrix}$$

Subtraction of matrices is performed by subtracting their elements term by term.

Example: Using the matrices A and B given above,

$$D = A - B = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 1 & 1 & -1 & -9 \\ 1 & 0 & 0 & -3 \end{bmatrix}$$

Multiplication of Matrices

The simplest type of multiplication involves matrices and scalars. The product of a scalar (which, as defined previously, is an ordinary number) and a matrix is a matrix whose elements are those of the original matrix, each multiplied by the scalar.

Example:

$$2 \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & 6 \end{bmatrix}$$

Multiplication of two matrices A and B can be performed <u>only if the number of columns of A equals the number of rows of B</u>. If A is an $(m \times n)$ matrix and B is an $(n \times p)$ matrix, the product E is an $(m \times p)$ matrix whose elements are the sums of certain products of the elements of A and B. The products involved can be most easily defined by considering some examples.

Example:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$E = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \end{bmatrix} \qquad \text{where}$$

$$e_{11} = a_{11}b_{11} + a_{12}b_{21}, \quad e_{12} = a_{11}b_{12} + a_{12}b_{22}, \quad e_{13} = a_{11}b_{13} + a_{12}b_{23},$$

$$e_{21} = a_{21}b_{11} + a_{22}b_{21}, \quad e_{22} = a_{21}b_{12} + a_{22}b_{22}, \quad e_{23} = a_{21}b_{13} + a_{22}b_{23}$$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$E = A \times B = \begin{bmatrix} -1 & 3 & 4 \\ 1 & 0 & -1 \end{bmatrix}$$

The elements of E are obtained in the following way:

$$e_{11} = (1 \times 1) + (2 \times -1) = -1, \; e_{12} = (1 \times 3) + (2 \times 0) = 3, \; e_{13} = (1 \times 2) + (2 \times 1) = 4$$

$$e_{21} = (0 \times 1) + (-1 \times -1) = 1, \; e_{22} = (0 \times 3) + (-1 \times 0) = 0, \; e_{23} = (0 \times 2) + (-1 \times 1) = -1$$

In general, e_{ij}, the ijth element of the matrix product E = AB equals the sum of the products of the elements of the ith row of A with the corresponding elements of the jth column of B, beginning at the left-hand side and top, respectively.

As a further illustration of matrix multiplication, consider the following:

$$A = \begin{bmatrix} 2.0613 & -0.3084 & 1.2149 \\ .1301 & 2.4359 & .5411 \\ 1.6210 & -.5041 & 2.1008 \\ .3825 & 1.7689 & -.4162 \end{bmatrix} \qquad B = \begin{array}{c} \\ \begin{bmatrix} 0.4268 & 1.3570 \\ -1.2243 & .6109 \\ .6064 & -.2184 \end{bmatrix} \end{array} \begin{array}{c} \Sigma \\ 1.7838 \\ -.6134 \\ .3880 \end{array}$$

The additional column to the right of the B matrix headed "Σ" is composed of row sums from the B matrix and is used to check the computation of E = AB.

$$E = AB = \begin{bmatrix} 1.9941 & 2.3434 \\ -2.5986 & 1.5465 \\ 2.5829 & 1.4329 \\ -2.2548 & 1.6906 \end{bmatrix} \begin{matrix} 4.3375\checkmark \\ -1.052\cancel{2}1 \\ 4.015\cancel{9}8 \\ .5642\checkmark \end{matrix}$$

If a_{ij} and b_{ij} are the elements of A and B, respectively, then e_{ij}, the element in the ith row and jth column of E, including the Σ column, is given by:

$$e_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j}$$

For example,

$$\begin{aligned} e_{32} &= a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\ &= (1.6210)(1.3570) + (-0.5041)(0.6109) \\ &\qquad + (2.1008)(-0.2184) \\ &= 1.4329 \end{aligned}$$

Similarly,

$$\begin{aligned} e_{1\Sigma} &= a_{11}b_{1\Sigma} + a_{12}b_{2\Sigma} + a_{13}b_{3\Sigma} \\ &= (2.0613)(1.7838) + (-0.3084)(-0.6134) \\ &\qquad + (1.2149)(0.3880) \\ &= 4.3375 \end{aligned}$$

If the sum across the ith row of E is identical with the element in the ith row of the Σ column of E, this is indicated by a check mark. This checks the computation of the ith row of E. Such was the case in the first and fourth rows. If there is a rounding error, that is, a discrepancy in the final decimal place, the sum across the row is used to "correct" the element in the Σ column. (See p. 5.) This was the case in the second and third rows.

It should be noted that these computations were carried out with 4 decimals instead of the 9 decimals used for most of the computations given in this handbook.

Unlike ordinary multiplication, the order in which matrix multiplication is performed is important; that is, AB does not necessarily equal BA. In fact, for the examples shown above, BA does not even exist. Only for square matrices is reverse multiplication possible and here AB in general is not the same as BA.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 4 & -1 \\ 10 & -3 \end{bmatrix} \quad B \times A = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$$

A row vector times a column vector equals a scalar, but a column vector times a row vector is a matrix.

Example: $F = (1 \quad 2 \quad -1) \quad G = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

$$F \times G = 5 \qquad G \times F = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 2 & 4 & -2 \end{bmatrix}$$

Therefore, it is important to perform multiplication in the indicated order.

Determinants

A matrix, as defined above, is an array of numbers; it has no value. Associated with a square matrix, however, is a numerical value called the determinant, written det. A = $|A|$.

The value of a second order determinant is defined as follows:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

The value of a third order determinant can be obtained as follows:

$$|B| = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = b_{11}(b_{22}b_{33} - b_{23}b_{32}) - b_{21}(b_{12}b_{33} - b_{13}b_{32}) + b_{31}(b_{12}b_{23} - b_{13}b_{22})$$

Example:

$$|B| = \begin{vmatrix} -6 & 11 & -8 \\ 2 & -3 & 0 \\ 2 & -5 & 4 \end{vmatrix} = (-6)\left[(-3 \times 4) - (0 \times -5)\right] - 2\left[(11 \times 4) - (-8 \times -5)\right] + 2\left[(11 \times 0) - (-8 \times -3)\right] = 16$$

Methods for evaluating determinants of higher order are given in the appendix, pp. 89, 101.

Inverse of a Matrix

The inverse of the square matrix A, designated as A^{-1}, is that matrix which when multiplied by A equals the unit matrix, that is,

$$A^{-1} A = I$$

Example:

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & 2 \end{bmatrix} \quad A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The inverse was obtained by application of the formula given below.

It should be emphasized that an inverse exists only for a square matrix, and is itself a square matrix. The concept of matrix inversion is analogous to division of ordinary numbers.

For a scalar, that is, a single number, the inverse is simply the reciprocal of that number.

For a second order matrix, the inverse can be obtained directly from the following formula:

$$A^{-1} = \begin{bmatrix} \frac{a_{22}}{|A|} & \frac{-a_{12}}{|A|} \\ \frac{-a_{21}}{|A|} & \frac{a_{11}}{|A|} \end{bmatrix}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and

$$|A| = a_{11}a_{22} - a_{21}a_{12}.$$

For a symmetrical matrix that has more than two rows and two columns, the inverse can be obtained by a variation of the Doolittle method. This method of matrix inversion has already been performed in the computation of the D matrix for the multiple regression problem. Throughout this handbook, most of the matrices that we invert are symmetrical, so that the inverse also is symmetrical To invert a symmetrical matrix by the variation of the Doolittle method, the worksheet is set up in the following form:

$$\begin{array}{ccccc c cccc c}
 & & & & & \Sigma m & & & & & \Sigma I \\
m_{11} & m_{12} & m_{13} & \cdots & m_{1k} & & 1 & 0 & 0 \ldots 0 & \\
 & m_{22} & m_{23} & \cdots & m_{2k} & & 0 & 1 & 0 \ldots 0 & \\
 & & m_{33} & \cdots & m_{3k} & & 0 & 0 & 1 \ldots 0 & \\
 & & & & \cdot & & \cdot & \cdot & \cdot \quad\; \cdot & \\
 & & & & \cdot & & \cdot & \cdot & \cdot \quad\; \cdot & \\
 & & & & \cdot & & \cdot & \cdot & \cdot \quad\; \cdot & \\
 & & & & m_{kk} & & 0 & 0 & 0 \ldots 1 &
\end{array}$$

The matrix on the left of order k is the one to be inverted; the one on the right is a unit matrix of order k. (Compare this with the upper section of table 3.) A forward solution is carried out as explained on p. 9. The inverse and its check are obtained in exactly the same way as the D matrix shown in table 3 and explained on p.11, since the D matrix is the inverse of the complete moment matrix. The inverse matrix is symmetrical and of order k.

Methods for inverting nonsymmetrical matrices are given in the appendix. (See p. 98.)

LIMITED INFORMATION APPROACH FOR SYSTEMS OF SIMULTANEOUS EQUATIONS

This section deals with computational methods for estimating structural coefficients and their standard errors in a system of simultaneous equations. The method used is called that of "maximum likelihood limited information single equation," commonly referred to as the "limited information" approach. In this method we proceed to estimate the coefficients and their standard errors for one equation at a time, with the simultaneity implied by the system taken into account in the computations. This is in contrast to a "full information" method, which solves simultaneously for all of the coefficients in all of the equations of the system. Although the full information method provides standard errors that are smaller (that is, statistically more efficient), the computations for most problems are much more difficult. For a description of full information methods, the reader is referred to Chernoff and Divinsky (2, pp. 252-259). Limited information estimates have the desirable statistical property of consistency and are as efficient as any other method that utilizes the same amount of information. 8/

In studies that deal with systems of simultaneous equations, it is convenient to divide the variables involved into two groups: (1) Those that are determined simultaneously within the system, commonly called "endogenous," and (2) those that affect the endogenous variables but are not directly affected by them, commonly called "predetermined." 9/ The endogenous variables are commonly designated by Y's and the predetermined variables by Z's. Frequently some other designation is used in the structural equations. For example, disposable income is at times designated by Y in structural equations, as in the problem on p. 29, but may be considered a predetermined variable. As with single-equation analyses, lower-case letters are used to indicate variables expressed as deviations from their respective means.

Certain other methods can be used to obtain statistical coefficients in systems of simultaneous equations that also have the desirable statistical property of consistency. One of these, the recursive approach, tends to be less efficient for most problems to which it is applicable than the limited information approach, and hence may be less desirable. Others, such as the method of instrumental variables and an approach suggested by Theil (19,20), give alternative estimates for the coefficients, depending on the particular variables used. The authors believe that most research analysts prefer methods that give unique answers, even though the answers obtained may be no better, in a statistical sense, than any one of the several alternative answers given by these other methods.

Criteria of Identification

Since this handbook is concerned primarily with computational methods, we give only a rule of thumb that relates to identification. One so-called "counting rule" establishes the conditions necessary for identifiability of an equation in a system of linear equations in which, as is commonly the case, the identifying information consists of a priori knowledge of which variables may actually enter each equation. This counting rule tells us that if the number of variables in the system (endogenous plus all predetermined variables, counted

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8/ For a brief discussion of the statistical properties efficiency and consistency, see Tintner (18, p. 86) and Klein (13, pp. 52-53).

9/ Predetermined variables include those determined outside of the system, commonly called "exogenous," and lagged values of the endogenous variables.

separately) minus the number of variables in the particular equation equals the number of endogenous variables in the system less one, we have a just identified equation. If the number of variables in the system minus the number of variables in a particular equation is greater than the number of endogenous variables in the system less one, we have an overidentified equation. Just identified and overidentified equations both are handled in this handbook essentially by the same approach, although slightly different steps are followed. If the number of variables in the system minus the number of variables in the particular equation is less than the number of endogenous variables in the system less one, we have an underidentified equation. An underidentified equation can not, in general, be fitted by statistical methods. 10/

This rule applies only when a single variable is multiplied by each structural coefficient. The following system serves as an example of one to which the counting rule does not apply. Here P_r and Q_f are endogenous and the other variables are assumed to be predetermined.

$$P_r = a_1 + b_{11}(Q_d + Q_f) + b_{12}Y$$

$$Q_f = a_2 + b_{21}(P_r - T) + b_{22}Y$$

A superficial application of the counting rule might suggest that these equations are each just identified, as there are 5 variables in the system, 2 of which are endogenous, and 4 variables appear in each equation. But each of these equations in fact is overidentified because we specify in effect that the coefficients for Q_d and Q_f and for P_r and T are respectively equal in absolute terms. This is indicated by the enclosure of each pair of variables in a parenthesis with a single common coefficient.

If an error is made in determining the degree of identification of an equation, it will be immediately apparent in the computation. If in doubt as to whether an equation is just identified, it is best to assume that it is overidentified, as errors are most likely to result from ignoring certain restrictions of the kind mentioned in the preceding example. As indicated in table 11, page 51, the computations involved in sections (1) to (8) are the same regardless of whether the equation is just identified or overidentified. If the assumption that the equation is overidentified is incorrect, but instead the equation is just identified, the computation involved in obtaining λ (section (9) when there are 2 endogenous variables in the equation and sections (9.1) and (10.1) when there are more than 2 endogenous variables) is indeterminate, that is, division by zero is indicated, except for rounding errors. If this occurs, the steps for a just identified equation should be followed. If, in fact, the equation is underidentified, indeterminate results will be obtained in attempting to apply the methods used for just identified equations.

For a further discussion of identification, the reader is referred to Koopmans (14) and Tintner (18, pp. 154-166).

10/ Certain other counting rules have been suggested. In general, these lead to the same conclusions as the rule given here. Methods based on the rank of certain matrices can be used to establish necessary and sufficient conditions of identifiability. The mathematics involved in these is beyond the scope of this handbook.

Computations Involved in a 2-Equation System

This example is taken from an unpublished study by Holland 11/ dealing with the demand-supply structure for construction lumber. The following variables for the years 1916-41 were used in fitting the equations:

Y_1 = price per thousand board feet of softwood construction lumber, in dollars

Y_2 = per capita shipments of softwood construction lumber, in board feet

Z_1 = per capita expenditures for new construction, including maintenance and repair, in dollars

Z_2 = per capita production of Portland cement, in barrels

Z_3 = index numbers of cost per thousand board feet of manufacturing softwood lumber, 1916-41 = 100

u_1 , u_2 = random error terms for equations (1) and (2), respectively.

The system can be written in the following form:

$$Y_1 = a_1 + b_{12}Y_2 + c_{11}Z_1 + c_{12}Z_2 \qquad (1)$$

$$Y_2 = a_2 + b_{21}Y_1 + c_{23}Z_3 + u_2 \qquad (2)$$

For computational purposes, however, it is convenient to rewrite the equations in the form:

$$b_1^1 y_1 + b_2^1 y_2 + c_1^1 z_1 + c_2^1 z_2 = u_1 \qquad (1.1)$$

$$b_1^2 y_2 + b_2^2 y_1 + c_1^2 z_3 = u_2 \qquad (2.1)$$

The superscripts indicate the equation. When working with any given equation, the superscripts are dropped. The computations are designed for an equation in which a linear combination of the variables is set equal to a random error term. Hence, all variables are put on the same side of the equality sign, in the same order as they appear in the original equation, and consecutively numbered b's and c's are assigned as coefficients to the y's and z's, respectively. Variables are expressed in terms of deviations from their respective means and hence are represented by lower case letters. The constant terms then equal zero and hence can be omitted. The relation between the two sets of coefficients in equations

11/ Holland, Israel Irving, Some Factors Affecting the Consumption of Lumber in the United States, with Emphasis on Demand. Ph.D. thesis, University of California, 1955. The authors and Mr. Holland realize that Z_2 should have been treated as an endogenous variable. A more complicated model that allows for this may be fitted later. The model as shown here, however, serves as a useful expository example.

(1), (2) and (1.1), (2.1) is as follows:

$1 = b_1^1$, $b_{12} = -b_2^1$, $c_{11} = -c_1^1$, $c_{12} = -c_2^1$; $1 = b_1^2$, $b_{21} = -b_2^2$, $c_{23} = c_1^2$.

After the computations are carried out, the values for the constants in the original equations are determined and the equations are rewritten in the form given in (1) and (2).

The counting rule is applied to one equation at a time to determine its degree of identification. For this study, the total number of variables in the system is 5 and the number of endogenous variables in the system is 2. For equation (1.1), the number of variables in the equation is 4. Applying the rule, $5-4 = 2-1$; therefore, the equation is just identified. For equation (2.1), the number of variables in the equation is 3; as 5-3 is greater than 2-1, the equation is overidentified. This information is used at a later stage.

Estimation of Coefficients.--In using the limited information approach, the coefficients of one equation at a time are estimated. But we first perform two preliminary computations which are applicable to all equations in the system. These are as follows:

(1) Compute adjusted augmented moments for all of the variables in the system. This computation is explained on p. 3. The worksheet should be set up so that the predetermined variables, the Z's, come first in numerical order, followed by the endogenous variables, the Y's, in numerical order. For this example, the computation of the augmented moments is shown in table 5, the adjustment factors in table 6, and the adjusted augmented moments in table 7. The adjusted augmented moments form the basis for all later computations, as they are the elements of the moment matrices. Table 7 is referred to as the moment matrix of all the variables in the system. Although we refer to this table in the explanations that follow, it is not necessary to set it up; we do so only for illustrative purposes. While learning the method, however, it may be convenient to use the equivalent of table 7. Later on, the parts shown in tables 8 to 10 may be written there directly. In the following explanations, a reference to table 7 indicates use of the adjusted augmented moments obtained by multiplying the augmented moments by their appropriate adjustment factors.

(2) Perform a forward Doolittle solution, using the variables shown in the diagram on p. 33. These computations first are indicated in a shorthand matrix notation. Some matrix terms are explained on p. 21. New notation is explained as needed.

We first set up the matrices M_{zz} and M_{zy}. M_{zz} includes the moments for all the z's in the system; M_{zy} includes moments of the z's on the y's for the system. Certain variables may at times be omitted from M_{zz} and M_{zy} (see p. 66 and 70). In this example, $z = (z_1, z_2, z_3)$ and $y = (y_1, y_2)$. Therefore, M_{zz} is a 3x3 matrix and M_{zy}, a 3x2 matrix. The elements of these matrices are the adjusted augmented moments, which can be obtained from table 7 or, if this is not used, directly from tables 5 and 6. Two Σ columns are obtained, one for M_{zz} and the other for M_{zy}. These columns consist of row sums for the respective matrices. As M_{zz} is a symmetrical matrix, terms below the diagonal are

Table 5.- Computation of augmented moments for lumber problem 1/

Item	Z_1	Z_2	Z_3	Y_1	Y_2	Σ
Sum	2,329.1	26.2	2,249.1	625.3	3,841.0	9,070.8 ✓
Mean	89.5	1.0	86.5	24.0	147.7	348.8 ✓
Extensions with--						
Z_1	228,371.8	2,541.8	203,041.3	58,380.1	363,086.7	855,421.9 ✓
	5,937,667.0	66,087.6	5,279,074.8	1,517,884.4	9,440,255.9	22,240,970.0 ✓
	5,424,893.1	61,070.0	5,238,468.7	1,456,504.4	8,946,226.7	21,127,163.1 ✓
	512,773.8	5,017.6	40,606.0	61,380.0	494,029.2	1,113,806.9 ✓
Z_2		28.5	2,267.9	647.7	4,067.6	9,553.7 ✓
		742.9	58,965.7	16,842.0	105,759.4	248,397.8 ✓
		687.4	58,971.4	16,396.4	100,711.0	237,836.3 ✓
		55.4	-5.6	445.6	5,048.4	10,561.4 ✓
Z_3			198,158.7	54,540.2	327,395.3	785,403.5 ✓
			5,152,126.9	1,418,045.9	8,512,279.8	20,420,493.4 ✓
			5,058,450.8	1,406,452.1	8,638,793.1	20,401,136.2 ✓
			93,676.1	11,593.7	-126,513.2	19,357.1 ✓
Y_1				15,895.1	95,718.1	225,181.4 ✓
				413,273.8	2,488,671.5	5,854,717.8 ✓
				391,050.1	2,401,930.9	5,672,334.0 ✓
				22,223.7	86,740.5	182,383.8 ✓
Y_2					611,975.6	1,402,243.6 ✓
					15,911,367.6	36,458,334.5 ✓
					14,753,281.0	34,840,942.8 ✓
					1,158,086.6	1,617,391.7 ✓

1/ Of the decimal places which were used in the original computations only one is shown in the table; therefore some of the computations may appear slightly in error. On the other hand, the check appears more accurate than is the case when more decimals are shown.

Table 6.- Adjustment factors for lumber problem

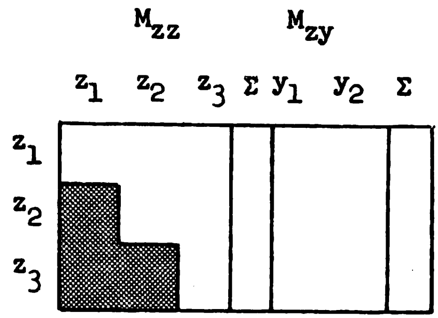
Variable	k_i	k_ik_j for -				
		z_1	z_2	z_3	y_1	y_2
z_1	0.001	0.000001	0.0001	0.00001	0.00001	0.000001
z_2	.1		.01	.001	.001	.0001
z_3	.01			.0001	.0001	.00001
y_1	.01				.0001	.00001
y_2	.001					.000001

Table 7.- Adjusted augmented moments for lumber problem 1/

Variable	z_1	z_2	z_3	y_1	y_2
z_1	0.5127	0.5017	0.4060	0.6138	0.4940
z_2		.5543	-.0056	.4456	.5048
z_3			9.3676	1.1593	-1.2651
y_1				2.2223	.8674
y_2					1.1580

1/ These computations were performed with 9 decimal places, of which only 4 appear in the table.

not written. But they must be included when obtaining the elements of the Σ column for M_{zz}. The worksheet is of the form:



A forward Doolittle solution is then carried out. Computations involved are explained on p. 9 and, for this example, they are shown in table 8. Note that in this diagram, as in those that follow, the omitted elements of symmetrical matrices are indicated by shaded boxes.

We now begin the computations for the single equations, working first with the overidentified equation (2.1). In the following computations, a new notation, y^* and z^*, is used. y^* is the vector of y's in the equation under consideration and in the order that they appear in the equation. z^* is defined in like manner. Therefore, in equation (2.1), $y^* = (y_2, y_1)$ and $z^* = (z_3)$. For equation (1.1), $y^* = (y_1, y_2)$ and $z^* = (z_1, z_2)$.

The first step is to compute $M_{y^*z}M_{zz}^{-1}M_{zy^*}$. Before beginning the actual computations, the analyst should determine the y^* that are needed in the several equations to be solved. This will provide a clue as to the form that the computation of $M_{y^*z}M_{zz}^{-1}M_{zy^*}$ should take. 12/ For the lumber problem the elements of $M_{y^*z}M_{zz}^{-1}M_{zy^*}$ are the same for each equation. Therefore it need be computed only once. The terms then can be rearranged for use with the second equation. In the present example, it is more convenient, computationally, to compute this matrix first for equation (1.1) and then rearrange the terms for use in connection with equation (2.1). If, however, a system had 8 endogenous variables and 4 equations, where for equation I, $y^* = (y_1, y_2)$; equation II, $y^* = (y_3, y_4)$; equation III, $y^* = (y_5, y_6)$; and equation IV, $y^* = (y_7, y_8)$, $M_{y^*z}M_{zz}^{-1}M_{zy^*}$ should be computed separately for each equation. This was the case for the system discussed on pp. 63-67. On the other hand, in the following system of equations where for equation I, $y^* = (y_1, y_6, y_7)$; equation II, $y^* = (y_2, y_3, y_5, y_6)$; equation III, $y^* = (y_2, y_3, y_5, y_7)$; and equation IV, $y^* = (y_1, y_2, y_3, y_4, y_5)$, the number of calculations could be substantially reduced by computing the complete $M_{yz}M_{zz}^{-1}M_{zy}$ rather than the separate $M_{y^*z}M_{zz}^{-1}M_{zy^*}$ for each equation.

The computation of $M_{y^*z}M_{zz}^{-1}M_{zy^*}$ for equation (1.1) is shown in table 8 immediately below the forward solution. The computation of this matrix from specified terms of the M_{zy} part of the forward solution is analogous to the computation of the D matrix for the multiple regression problem (see p. 11). Using only the last two rows in each section of the forward solution, the element in the ith row and jth column of $M_{y^*z}M_{zz}^{-1}M_{zy^*}$ is obtained by cumulating the

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12/ The method used in this handbook differs from that described in Chernoff and Divinsky (2, p. 242) where the complete $M_{yz}M_{zz}^{-1}M_{zy}$ is always computed. For some types of equations this involves many needless computations

Table 8.--Computations applicable to entire system for lumber problem 1/

Forward solution

Row	M_{zz} z_1	M_{zz} z_2	M_{zz} z_3	Σ_z	M_{zy} y_1	M_{zy} y_2	Σ_y
(1) z_1	0.5127	0.5017	0.4060	1.4205	0.6138	0.4940	1.1078
(2) z_2		.5543	- .0056	1.0504	.4456	.5048	.9504
(3) z_3			9.3676	9.7680	1.1593	-1.2651	- .1057
(1)	0.5127	0.5017	0.4060	1.4205	0.6138	0.4940	1.1078
(1")	1.	.9785	.7918	2.7704 ✓	1.1970	.9634	2.1604 ✓
(2)		.5543	- .0056	1.0504	.4456	.5048	.9504
(1)(-0.9785)		- .4909	- .3973	-1.3900	- .6006	- .4834	-1.0840
(2')		.0633	- .4030	- .3396 ✓	- .1549	.0214	- .1335 ✓
(2")		1.	-6.3653	-5.3653 ✓	-2.4481	.3383	-2.1098 ✓
(3)			9.3676	9.7680	1.1593	-1.2651	- .1057
(1)(-0.7918)			- .3215	-1.1249	- .4860	- .3912	- .8772
(2')(6.3653)			-2.5652	-2.1622	- .9866	.1363	- .8502
(3')			6.4807	6.4807 ✓	- .3132	-1.5200	-1.8332 ✓
(3")			1.	1. ✓	- .0483	- .2345	- .2828 ✓

Computation of $M_{y*z} M_{zz}^{-1} M_{zy*}$ for equation (1.1)

	y_1	y_2	Σ
y_1	1.1293	0.6124	1.7417 ✓
y_2		.8397	1.4521 ✓

$M_{y*z} M_{zz}^{-1} M_{zy*}$ for equation (2.1)

	y_2	y_1	Σ
y_2	0.8397	0.6124	1.4521 ✓
y_1		1.1293	1.7417 ✓

1/ These computations were performed with 9 decimal places, of which only 4 appear in the table; therefore, some of the computations may appear slightly in error.

products of the terms in the next to last row of the ith column with the terms in the last row of the jth column. For example, if m_{ij} are the elements of $M_{y*z}M_{zz}^{-1}M_{zy*}$, then:

$$m_{11}=(0.6138)(1.1970)+(-0.1549)(-2.4481)+(-0.3132)(-0.0483)=1.1293$$

$$m_{12}=(0.6138)(0.9634)+(-0.1549)(0.3383)+(-0.3132)(-0.2345)=0.6124$$

$$m_{1\Sigma}=(0.6138)(2.1604)+(-0.1549)(-2.1098)+(-0.3132)(-0.2828)=1.7417$$

The check placed next to the items in the Σ column at the bottom of this section indicates that the computations are correct, except for possible rounding errors. As this matrix is symmetrical, the terms below the main diagonal are not computed. In obtaining the check by summing across the items in the row, the omitted terms must be included in the sum. $M_{y*z}M_{zz}^{-1}M_{zy*}$ for equation (2.1) is obtained by rearranging the elements of that for equation (1.1). The nature of this rearrangement is clear from the table.

In the following computations, we first work with the overidentified equation (2.1). This is done chiefly for expository reasons, as it is easier to understand the computations for an overidentified equation than for a just identified one. In the solution of an actual problem, the equations probably would be fitted in order. In some systems of equations, certain equations may involve only a single endogenous variable. In such cases, these equations can be fitted directly by least squares, but the predetermined variables involved in them should be included in the M_{zz} matrix. Equations that can be handled by least squares always can be fitted by the method described in the first major section of this handbook. (See p. 2.) In some instances, however, many computations can be saved by what in effect is a simultaneous approach for both the equations that are to be fitted by the limited information method and those that can be handled by least squares. Examples are discussed in detail on pp. 69-76.

In operations that involve multiplication of symmetrical matrices that have missing elements, the missing elements should be filled in before performing the multiplication; otherwise it is difficult to determine which elements are involved in the product. Elements can be omitted from the product matrix, however, unless it is to be used in further multiplications, as it is always symmetrical. Missing elements are not needed in connection with addition and subtraction, as this is carried out on a term by term basis. Missing elements that are needed have been shown in tables 9 and 10. This explains why some matrices are shown in full, while in others the elements below the main diagonal are omitted.

In obtaining the coefficients for equation (2.1) we first compute $W_{y*y*} = M_{y*y*} - M_{y*z}M_{zz}^{-1}M_{zy*}$. This computation is of the form:

Table 9.- Limited information computations for equation (2.1) in the lumber problem (an overidentified equation) 1/

(2) M_{y*y*}

	y_2	y_1	Σ
y_2 :	1.1580	0.8674	2.0254
y_1 :		2.2223	3.0897

(1) $M_{y*z}M_{zz}^{-1}M_{zy*}$

	y_2	y_1	Σ
y_2 :	0.8397	0.6124	1.4521 ✓
y_1 :		1.1293	1.7417 ✓

(3) W_{y*y*}

	y_2	y_1	Σ
y_2 :	0.3183	0.2550	0.5733 ✓
y_1 :		1.0930	1.3480 ✓

(4) M_{z*z*}^{-1}

	z_3
z_3 :	0.1067

(5) M_{z*y*}

	y_2	y_1	Σ
z_3 :	-1.2651	1.1593	-0.1057

(6) P'

	y_2	y_1	Σ
z_3 :	-0.1350	0.1237	-0.0112 ✓

(7) $M_{y*z*}P'$

	y_2	y_1	Σ
y_2 :	0.1708	-0.1565	0.0142 ✓
y_1 :		.1434	- .0130 ✓

(9)
$p_1 = 0.2829 \quad p_3 = 0.0680$
$p_2 = .6527 \quad 2p_3 = .1361$
$p_2^2 - 4p_1p_3 = 0.3490$

$\sqrt{p_2^2-4p_1p_3} = 0.5908$

$$\lambda = \frac{p_2 + \sqrt{p_2^2-4p_1p_3}}{2p_3} = 9.1371$$

(10) λB_{y*y*}

	y_2	y_1	Σ
y_2 :	6.1114	7.0263	13.1378 ✓
y_1 :		9.0077	16.0341 ✓

(11) R_{y*y*}

	y_2	y_1	Σ
y_2 :	-5.7931	-6.7713	-12.5644 ✓
y_1 :		-7.9147	-14.6860 ✓

(12) b'

y_2 :	1.
y_1 :	- .8555

(13) c'

z_3 :	0.2409

(8) B_{y*y*}

	y_2	y_1	Σ
y_2 :	0.6688	0.7689	1.4378 ✓
y_1 :		.9858	1.7548 ✓

(19)
$D = \frac{1}{\lambda} = 0.1094$
$\frac{D}{bWb'} = 0.1604$
$1+D = 1.1094$
$C = (1+D)\ bWb' = 0.7567$
$C^* = \frac{C}{26-3} = 0.0329$

(14) b

y_2	y_1
1.	-0.8555

(15) W_{y*y*}

	y_2	y_1	Σ
y_2 :	0.3183	0.2550	0.5733 ✓
y_1 :	.2550	1.0930	1.3480 ✓

(16) bW

y_2	y_1	Σ
0.1002	-0.6801	-0.5799 ✓

(17) bWb'

0.6820

(18) $_{11}[(bW)'\ bW]$

	y_1
y_1 :	0.4625

(21) $_{11}B$

	y_1
y_1 :	0.9858

(20) $_{11}G$

	y_1
y_1 :	0.0742

(22) $_{11}H$

	y_1
y_1 :	0.9116

(24) $_{01}P'$

	y_1
z_3 :	0.1237

(23) F_{bb}

	y_1
y_1 :	1.0969

(25) F'_{bc}

	y_1
z_3 :	0.1357

(26) U

	z_3
z_3 :	0.0168

(27) M_{z*z*}^{-1}

	z_3
z_3 :	0.1067

(28) F_{cc}

	z_3
z_3 :	0.1235

(29) C^*F_{bb}

	y_1
y_1 :	0.0360

(30) C^*F_{cc}

	z_3
z_3 :	0.0040

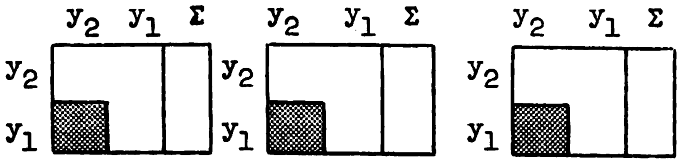
(1)	(2) Coef.	(3) S. E.	(4) k_i	(5) $k_i' = \frac{k_i}{k_{Y_2}}$	(6) Deadj. coef.	(7) Deadj. S.E.	(8) Deadj. mean	(9) Deadj. mean x deadj. coef.
Y_2 :	b_1 = 1.	-	0.001	-	-	-	147.7307	-
Y_1 :	b_2 = .8555	0.1899	.01	10.	8.5553	1.8997	24.0515	205.7689
Z_3 :	c_1 = -.2409	.0637	.01	10.	-2.4093	.6375	86.5038	-208.4210
								-2.6520

$a = 147.7307 - (-2.6520) = 150.3828$

$$Y_2 = 150.3828 + \underset{(1.8997)}{8.5553}\ Y_1 - \underset{(0.6375)}{2.4093}\ Z_3$$

1/ These computations were performed with 9 decimal places of which only 4 appear in the table; therefore some of the computations may appear to be slightly in error.

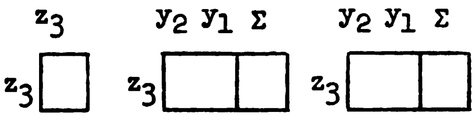
$$M_{y^*y^*} \quad - M_{y^*z}M_{zz}^{-1}M_{zy^*} \quad = \quad W_{y^*y^*}$$



These computations are shown in sections (1), (2) and (3) of table 9. Section numbers refer to the order in which the computations are performed. Arranging them as shown avoids needless copying of data. Steps involved in these 3 sections are as follows: (1) Copy $M_{y^*z}M_{zz}^{-1}M_{zy^*}$ for equation (2.1) from table 8. Here, as in other instances of copying matrices, the Σ column is also copied. The check is obtained by recomputing the sum across the copied row, including the omitted figures. This should be identical with the figure in the Σ column. 13/ If not, an error was made in copying. (2) $M_{y^*y^*}$, a symmetrical moment matrix of the y*'s, is obtained from table 7. For this example the y's are reversed in order. A column, Σ, composed of row sums is computed. (3) $W_{y^*y^*}$ is obtained by subtracting (1) from (2), element by element, including the items in the Σ columns. The sums across the rows of $W_{y^*y^*}$ are obtained and the fact that they are identical with the items in the Σ column of the respective rows of $W_{y^*y^*}$ is indicated by a check mark. This provides a check on the computation.

We now compute $P' = M_{z^*z^*}^{-1}M_{z^*y^*}$. This computation is of the form:

$$M_{z^*z^*}^{-1} \quad X \quad M_{z^*y^*} \quad = \quad P'$$



See sections (4), (5), and (6) of table 9. (4) z* has only one element. Therefore, $M_{z^*z^*}$ is a scalar or ordinary number equal to 9.3676. Its inverse, $M_{z^*z^*}^{-1}$, is simply the reciprocal of that number or 1/9.3676 = 0.1067. If z* had more than one element, other methods described on p. 26 would be used for inverting $M_{z^*z^*}$ (5) $M_{z^*y^*}$ is a moment matrix. It can be copied from the upper section of the forward Doolittle solution, table 8, or directly from table 7. A column, Σ, composed of row sums, is computed. (6) As explained in the section on matrix multiplication, a scalar times a matrix equals a matrix whose elements are those of the original matrix, including Σ, each multiplied by the scalar. That the sum across the row of P' is identical (except for possible rounding error) with the element in the Σ column of P' is indicated by a check mark.

We next compute $M_{y^*z^*}P'$. See sections (5), (6), and (7) of table 9. Sections (5) and (6) were obtained in the previous step. (7) To obtain m_{ij}, the ijth element of $M_{y^*z^*}P'$, multiply the ith column of $M_{z^*y^*}$, excluding Σ,

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13/ Rounding errors occur only when the indicated operation is multiplication or division.

with the jth column of P', including Σ. For example:

$$m_{11} = (-1.2651)(-0.1350) = 0.1708$$

$$m_{12} = (-1.2651)(0.1237) = -0.1565$$

$$m_{1\Sigma} = (-1.2651)(-0.0112) = 0.0142$$

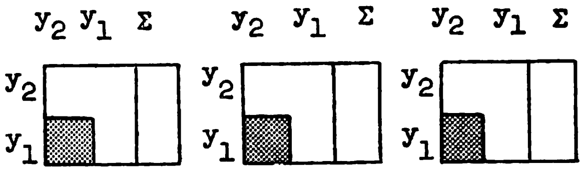
$$m_{22} = (1.1593)(0.1237) = 0.1434$$

$$m_{2\Sigma} = (1.1593)(-0.0112) = -0.0130.$$

That the sums across the rows of $M_{y*z*}P'$ are identical (except for possible rounding errors) with the elements in the respective rows of the Σ column of $M_{y*z*}P'$ is indicated by a check mark. Use of column-by-column multiplication is discussed on p. 101. As the product matrix is symmetrical, elements below the diagonal need not be computed.

We now compute $B_{y*y*} = M_{y*z}M_{zz}^{-1}M_{zy*} - M_{y*z*}P'$. This computation is of the form:

$$M_{y*z}M_{zz}^{-1}M_{zy*} \quad - \quad M_{y*z*}P' \quad = \quad B_{y*y*}$$



See section (1), (7) and (8) of table 9. Sections (1) and (7) were obtained in previous steps. (8) B_{y*y*} is obtained by subtracting (7) from (1), element by element, including the elements in the Σ columns. The sums across the rows of B_{y*y*} are obtained and the fact that they are identical (except for errors that result from the carrying of only 4 decimals) with the elements in the Σ column of the respective rows of B_{y*y*} is indicated by a check mark.

We now compute λ. See section (9) of table 9. Let w_{ij} be the elements of W_{y*y*} in section (3), and b_{ij}, the elements of B_{y*y*} in section (8). Then compute:

$$p_1 = |W| = w_{11}w_{22} - w_{12}^2$$

$$p_2 = w_{11}b_{22} + b_{11}w_{22} - 2b_{12}w_{12}$$

$$p_3 = |B| = b_{11}b_{22} - b_{12}^2$$

$$1/2p_3$$

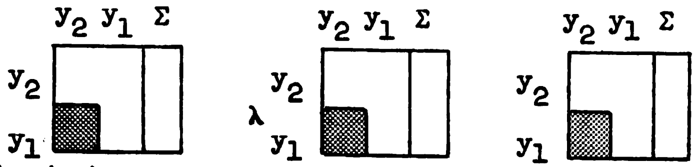
$$p_2^2 - 4p_1p_3$$

$$\sqrt{p_2^2 - 4p_1p_3}$$

$$\lambda = \frac{1}{2p_3} \left(p_2 + \sqrt{p_2^2 - 4p_1p_3}\right)$$

We next compute $R_{y*y*} = W_{y*y*} - \lambda B_{y*y*}$. This computation is of the form:

$$W_{y*y*} \quad - \quad \lambda B_{y*y*} \quad = \quad R_{y*y*}$$



See sections (3), (10), and (11) of table 9. Section (3) was obtained previously. Section (10) is obtained by multiplying each element of B_{y*y*}, including Σ, obtained in section (8), by the scalar, λ, computed in section (9). That the sums across the rows of λB_{y*y*} are identical (except for possible rounding errors) with the elements in the respective rows of the Σ column of λB_{y*y*} is indicated by check marks. (11) R_{y*y*} is obtained by subtracting (10) from (3), element by element, including the elements in the Σ columns. The sums across the rows of R_{y*y*} are obtained and the fact that they are identical (except for errors that result from the carrying of only 4 decimals) with the elements in the Σ column of the respective rows by R_{y*y*} is indicated by check marks. An additional check on the computation of R_{y*y*}, as well as on the computation of λ, is to compute $|R| = r_{11}r_{22} - r_{12}^2$, where the r_{ij} are elements of R_{y*y*}. $|R|$ should equal zero.

We now compute:

$$b' = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

b' is the vector of the coefficients of the endogenous variables, the $y*$'s. For equation (2.1), and any other equation where the number of endogenous variables is equal to two:

$$b' = \begin{bmatrix} 1 \\ \dfrac{-r_{11}}{r_{12}} \end{bmatrix}$$

See section (12) of table 9. $b_1 = 1$ by definition (see p. 31) and $b_2 = -\dfrac{r_{11}}{r_{12}}$, where r_{11} and r_{12} are elements of R_{y*y*}, obtained in section (11). 14/

We next compute $c' = -P'b'$. c' is the vector of the coefficients of the predetermined variables, the $z*$'s. This computation is of the form:

14/ This method for obtaining the coefficients, already normalized, differs from that discussed in Klein (13, pp. 179-180) and Chernoff and Divinsky (2, p. 245), where the coefficients are first obtained and then normalized. Normalization, in this sense, refers to the transformation of coefficients so that $b_1 = 1$.

$$-P' \quad X \quad b' \quad = \quad c'$$

$$\begin{array}{c|cc|} & y_2 & y_1 \\ \hline -z_3 & & \\ \hline \end{array} \qquad \begin{array}{c|c|} \hline y_2 & 1 \\ y_1 & \\ \hline \end{array} \qquad \begin{array}{c|c|} \hline z_3 & \\ \hline \end{array}$$

Note that for equation (2.1), z^* has only one element, and therefore c' is a scalar. See sections (6), (12), and (13) of table 9. Sections (6) and (12) were obtained previously. (13) c' is obtained by multiplying $-P'$, excluding Σ, by b'. Computationwise, it is easier to neglect the minus sign prefixed to P'. Instead, multiply P' by b', and then change the sign of the result. This gives c'. The check in this step is one of recomputation.

In an actual problem, we would continue with equation (2.1) and estimate the standard errors of the coefficients just computed. But for illustrative purposes, we first proceed to estimate the coefficients of the just identified equation (1.1). The relevant computations are shown in table 10.

For equation (1.1), $z^* = (z_1, z_2)$ and $y^* = (y_1, y_2)$. The matrix shown in section (1), table 10, was computed in table 8. Since the same y's, although in reverse order, are involved in equation (2.1), some of the computations can be eliminated. This always is the case when the same variables appear in several equations. For example, compare the matrices in sections (2) and (3) of tables 9 and 10. Those of table 10 can be obtained from those in table 9 by reversing the order of the y's. In making the computations for several equations, steps should be copied whenever possible. The copying can be checked as described on page 37. After copying these matrices, we compute $P' = M^{-1}_{z^*z^*}M_{z^*y^*}$. This computation is of the form:

$$M^{-1}_{z^*z^*} \quad X \quad M_{z^*y^*} \quad = \quad P'$$

$$\begin{array}{c|cc|} & z_1 & z_2 \\ \hline z_1 & & \\ z_2 & & \\ \hline \end{array} \qquad \begin{array}{c|cc|c|} & y_1 & y_2 & \Sigma \\ \hline z_1 & & & \\ z_2 & & & \\ \hline \end{array} \qquad \begin{array}{c|cc|c|} & y_1 & y_2 & \Sigma \\ \hline z_1 & & & \\ z_2 & & & \\ \hline \end{array}$$

See section (4), (5), and (6) of table 10. (4) Since $M_{z^*z^*}$ is a 2x2 matrix, it should be inverted according to the definition given on page 26. These computations, shown in sections (4a) and (4b), are as follows: (4a) Write $M_{z^*z^*}$, which is obtained from the upper section of the forward Doolittle solution in table 8. A column, Σ, composed of row sums, is also computed but does not enter into the computations until a later stage. (4b) Write the adjoint of $M_{z^*z^*}$. This is obtained by interchanging the elements in the main diagonal of $M_{z^*z^*}$ and changing the sign of the elements not in the main diagonal. Preceeding this matrix, write $1/|M|$ where $|M|$ is the product of the diagonal elements of $M_{z^*z^*}$ minus the product of the nondiagonal elements.

Table 10.- Limited information computations for equation (1.1) in the lumber problem (a just identified equation) 1/

(4a) $M_{z^*z^*}$

	z_1	z_2	Σ
	0.5127	0.5017	1.0145
		.5543	1.0560

(4b) adj. $M_{z^*z^*}$

$$\frac{1}{0.0324}\begin{bmatrix} 0.5543 & -0.5017 \\ & .5127 \end{bmatrix}$$

(2) $M_{y^*y^*}$

	y_1	y_2	Σ
y_1	2.2223	0.8674	3.0897 ✓
y_2		1.1580	2.0254 ✓

(1) $M_{y^*z}M_{zz}^{-1}M_{zy^*}$

	y_1	y_2	Σ
y_1	1.1293	0.6124	1.7417 ✓
y_2		.8397	1.4521 ✓

(3) $W_{y^*y^*}$

	y_1	y_2	Σ
y_1	1.0930	0.2550	1.3480 ✓
y_2		.3183	.5733 ✓

$M_{z^*z^*}^{-1}$

	z_1	z_2	Σ
	17.0739	-15.4556	1.6183
	-15.4556	15.7947	.3391

(5) $M_{z^*y^*}$

	y_1	y_2	Σ
z_1	0.6138	0.4940	1.1078
z_2	.4456	.5048	.9504

(6) P'

	y_1	y_2	Σ
z_1	3.5925	0.6323	4.2249 ✓
z_2	-2.4481	.3383	-2.1098 ✓
Σ'	1.1444	.9707	

(7) $M_{y^*z^*}P'$

	y_1	y_2	Σ
y_1	1.1141	0.5389	1.6531 ✓
y_2		.4832	1.0221 ✓

(12.2) b'

y_1	1.
y_2	-0.2061

(13) c'

z_1	-3.4622
z_2	2.5178
Σ'	-0.9443 ✓

(8) $B_{y^*y^*}$

	y_1	y_2	Σ
y_1	0.0151	0.0734	0.0886 ✓
y_2		.3565	.4299 ✓

(9.2) $_{11}B^{-1}$

y_2	2.8050

(10.2) B_1

y_2	0.0734

(11.2) $_1b'$

y_2	-0.2061

(14) b

	y_1	y_2
	1.	-0.2061

(15) $W_{y^*y^*}$

	y_1	y_2	Σ
y_1	1.0930	0.2550	1.3480 ✓
y_2	.2550	.3183	.5733 ✓

(24) $_{01}P'$

	y_2
z_1	0.6323
z_2	.3383
Σ'	.9707 ✓

(23.2) F_{bb}

	y_2
y_2	2.8050

(25) F'_{bc}

	y_2
z_1	1.7738
z_2	.9490
Σ'	2.7228 ✓

(26) U

	z_1	z_2	Σ
z_1	1.1217	0.6001	1.7218 ✓
z_2		.3210	.9212 ✓

(27) $M_{z^*z^*}^{-1}$

	z_1	z_2	Σ
z_1	17.0739	-15.4556	1.6183 ✓
z_2		15.7947	.3391 ✓

(16) bW

	y_1	y_2	Σ
	1.0404	0.1893	1.2298 ✓

(17) bWb'

	1.0014

(29) C^*F_{bb}

	y_2
y_2	0.1276

(28) F_{cc}

	z_1	z_2	Σ
z_1	18.1957	-14.8554	3.3402 ✓
z_2		16.1158	1.2603 ✓

(30) C^*F_{cc}

	z_1	z_2	Σ
z_1	0.8282	-0.6762	0.1520 ✓
z_2		.7335	.0573 ✓

(19.2) $C^* = \frac{bWb'}{26-4} = 0.0455$

(1)	(2) Coef.	(3) S. E.	(4) k_i	(5) $k'_i = \frac{k_i}{k_{Y_1}}$	(6) Deadj. coef.	(7) Deadj. S.E.	(8) Deadj. mean	(9) Deadj. mean x deadj. coef.
Y_1	$b_1 = 1.$	-	0.01	-	-	-	24.0515	-
Y_2	$b_2 = .2061$	0.3573	.001	.1	.0206	0.0357	147.7307	3.0449
Z_1	$c_1 = 3.4622$	.9100	.001	.1	.3462	.0910	89.5823	31.0155
Z_2	$c_2 = -2.5178$	.8565	.1	10.0	-25.1786	8.5650	1.0084	-25.3916
								8.6688

a = 24.0515 -8.6688 = 15.3826

$$\underset{}{Y_1 = 15.3826 + \underset{(.0357)}{0.0206}\,Y_2 + \underset{(.0910)}{0.3462}\,Z_1 \underset{(8.5650)}{-25.1786}\,Z_2}$$

1/ These computations were performed with 9 decimal places of which only 4 appear in the table; therefore some of the computations may appear to be slightly in error.

In this example, $|M| = (0.5127)\ (0.5543) - (0.5017)\ (0.5017) = 0.0324$. (4) M^{-1}_{z*z*} is obtained by multiplying the elements of the adjoint of M_{z*z*} by $1/M$. This computation is the same as that described on page 26, although in the latter no mention is made of the adjoint matrix. Insert the missing elements and compute a column, Σ, composed of row sums, for M^{-1}_{z*z*}. The check on the computation of the inverse is obtained by cumulatively multiplying the elements in the Σ column of M_{z*z*} by the elements in the Σ column of the corresponding rows of M^{-1}_{z*z*}; this sum should approximately 15/ equal two, the order of the matrices involved. (5) M_{z*y*} is obtained from the upper section of the forward Doolittle solution, table 8. A column, Σ, composed of row sums, is computed. (6) P' is obtained by multiplying M^{-1}_{z*z*} by M_{z*y*}. As explained in the section on matrix multiplication, the ijth element of P' is obtained by summing the products of the elements of the row of M^{-1}_{z*z*}, excluding Σ, with the elements of the jth column of M_{z*y*}, including the Σ column. If p'_{ij} are the elements of P', then:

$$p'_{11} = (17.0739)\ (0.6138) + (-15.4556)\ (0.4456) = 3.5925$$

$$p'_{12} = (17.0739)\ (0.4940) + (-15.4556)\ (0.5048) = 0.6323$$

$$p'_{1\Sigma} = (17.0739)\ (1.1078) + (-15.4556)\ (0.9504) = 4.2249$$

$$p'_{21} = (-15.4556)(0.6138) + (15.7947)\ (0.4456) = -2.4481$$

$$p'_{22} = (-15.4556)(0.4940) + (15.7947)\ (0.5048) = 0.3383$$

$$p'_{2\Sigma} = (-15.4556)(1.1078) + (-15.7947)\ (0.9504) = -2.1098$$

That the sums across the rows of P' are identical (except for possible rounding errors) with the elements in the respective rows of the Σ column of P' is indicated by check marks. This checks the computation. An additional row, Σ', composed of column sums, is computed for P' and used in later computations.

We now compute $M_{y*z*}P'$. See sections (5), (6), and (7) of table 10. Sections (5) and (6) were obtained in previous steps. (7) The ijth element of $M_{y*z*}P'$ is obtained by summing the products of the elements of the ith column of M_{z*y*}, excluding Σ, with the elements in the jth column of P', including Σ but excluding Σ'. If m_{ij} are the elements of $M_{y*z*}P'$, then:

15/ The word "approximately" is used deliberately in preference to the term "except for possible rounding errors" which has been used in connection with checks on matrix operations. In an overall check of this type, an extra accumulation of rounding errors occurs which has to be taken into account.

$$m_{11} = (0.6138)\ (3.5925) + (0.4456)\ (-2.4481) = 1.1141$$

$$m_{12} = (0.6138)\ (0.6323) + (0.4456)\ (0.3383) = 0.5389$$

$$m_{1\Sigma} = (0.6138)\ (4.2249) + (0.4456)\ (-2.1098) = 1.6531$$

$$m_{22} = (0.4940)\ (0.6323) + (0.5048)\ (0.3383) = 0.4832$$

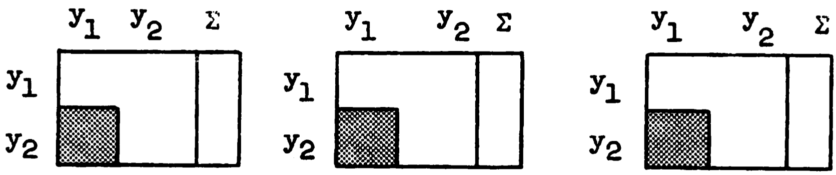
$$m_{2\Sigma} = (0.4940)\ (4.2249) + (0.5048)\ (-2.1098) = 1.0221$$

That the sums across the rows of $M_{y*z*}P'$ are identical (except for possible rounding errors) with the elements in the respective rows of the Σ column of $M_{y*z*}P'$ is indicated by check marks. It will be remembered that column-by-column multiplication was used in the corresponding step in table 9.

Although the matrices in sections (5) and (6) are nonsymmetrical, the matrix in section (7) is symmetrical. Hence, in obtaining the product matrix P', all elements must be computed, but in obtaining the product matrix $M_{y*z*}P'$, the elements below the main diagonal need not be computed.

We next compute $B_{y*y*} = M_{y*z}M_{zz}^{-1}M_{zy*} - M_{y*z*}P'$. This computation is of the form:

$$M_{y*z}M_{zz}^{-1}M_{zy*} \quad - \quad M_{y*z*}P' \quad = \quad B_{y*y*}$$

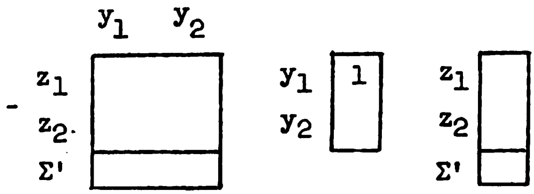


See sections (1), (7), and (8) of table 10. Sections (1) and (7) were obtained in previous steps. (8) Computations involved in obtaining B_{y*y*} are explained on page 35.

We now compute ${}_1b' = - \left[{}_{11}B\right]^{-1}B_1$. b' is the vector of the regression coefficients of the endogenous variables, the y's, in equation (1.1), that is, $b' = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, but $b_1 = 1$ by definition (see page 31). Therefore $b' = \begin{bmatrix} 1 \\ b_2 \end{bmatrix}$. ${}_1b'$ is b' with its first element omitted, that is, ${}_1b' = \left[b_2\right]$ ${}_{11}B$ is B_{y*y*} with the first row and first column omitted. B_1 is the first column of B_{y*y*} with the first element omitted. See sections (9.2), (10.2), and (11.2) of table 10. (9.2) We copy B_{y*y*}, omitting the first row and first column. B_{y*y*} was obtained in section (8). ${}_{11}B$ is therefore a single element, 0.3565; its inverse, as explained previously, is the reciprocal of that element, so that ${}_{11}B^{-1} = 1/0.3565 = 2.8050$. (10.2) Before obtaining B_1, missing elements in B_{y*y*} must be inserted, at least mentally. The first column of B_{y*y*} with its first element omitted is a scalar, 0.0734. (11.2) ${}_1b'$ is obtained by multiplying $- \left[{}_{11}B\right]^{-1}$ by B_1. ${}_1b'$ is the product of two scalars and is itself a scalar. The check here is one of recomputation. As in section 13 for equation (2.1), computationwise, it is easier to neglect the minus sign prefixed to ${}_{11}B^{-1}$ Instead multiply ${}_{11}B^{-1}$ by B_1. Then change the sign of the result to get ${}_1b'$. If more than two endogenous variables were included in the equation, B_{y*y*} would be of an order greater than two and other methods would be used to invert ${}_{11}B$ (see p. 26). ${}_1b'$ would then be the product of a matrix times a vector and would itself be a vector.

We now compute $c' = -P'\,b'$. This computation is of the form:

$$-P' \quad X \quad b' \quad = \quad c'$$

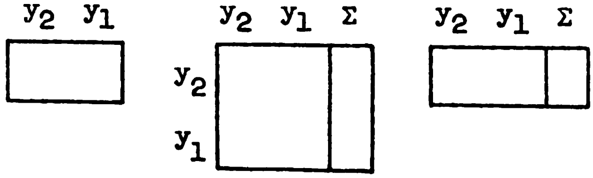


See sections (6), (12.2), and (13) of table 10. Section (6) was obtained previously; in the following computation we make use of Σ', a column sum, rather than Σ, a row sum. (12.2) ${}_1b'$, which is b' with its first element omitted, was obtained in section (11.2). However, $b_1 = 1$ by definition. Therefore we can write b' by inserting the omitted element, b_1, in ${}_1b'$. (13) c' is obtained by multiplying $-P'$, including Σ' and excluding Σ, by b'. That the sum of the elements of c' is identical (except for possible rounding error) with the element in the Σ' row of c' is indicated by a check mark. This checks the computation. Computationwise, it is easier to neglect the minus sign prefixed to P'. Instead, multiply P', including Σ' and excluding Σ, by b' and change the sign of the result to get c', including Σ'. Then perform the check in the usual way.

Estimation of Standard Errors of the Coefficients.--We now consider the computations involved in obtaining the standard errors of the coefficients, beginning with the overidentified equation (2.1) (see table 9).

First we compute bW_{y*y*}. This computation is of the form:

$$b \quad X \quad W_{y*y*} \quad = \quad bW_{y*y*}$$



Section numbers are continuations of those used to obtain the coefficients. See sections (14), (15), and (16) of table 9. (14) b is a row vector; it is the transpose of the column vector b' obtained in section (12). (15) W_{y*y*}

was computed in section (3). It is copied in section (15), including the missing element and Σ. The check marks next to the items in the Σ column indicate that they are identical with the sums across the rows of the copied W_{y*y*}. They confirm the copying. If preferred, this computation can be carried out directly by use of the data in section (3) including the missing element. (16) bW_{y*y*} is obtained by multiplying b by W_{y*y*}, including Σ. That the sum across the row of bW_{y*y*} is identical (except for possible rounding error) with the item in the Σ column is indicated by a check mark. The reader will note that subscript designations for W_{y*y*} are omitted in sections (16) to (18) of the table.

We now compute $bW_{y*y*}\, b'$. See section (14), (16) and (17) of table 9. Sections (14) and (16) were obtained previously. (17) $bW_{y*y*}\, b'$ is obtained by cumulating the product of the ith element of bW_{y*y*}, excluding Σ, with the ith element of b. (bW_{y*y*} and b are always vectors and therefore $bW_{y*y*}\, b'$ is always a scalar.) In this example, $bW_{y*y*}\, b' = (0.1002)\,(1) + (-0.6801)(-0.8555) = 0.6820$. This is, in effect, row-by-row multiplication (see p.101).

We next compute $_{11}\{(bW_{y*y*})'bW_{y*y*}\}$. See sections (16) and (18) of table 9. Section (16) was obtained previously. (18) $_{11}\{(bW_{y*y*})'bW_{y*y*}\}$ is the matrix $(bW_{y*y*})'bW_{y*y*}$ with its first row and first column omitted. We do not have to compute the entire matrix, $(bW_{y*y*})'bW_{y*y*}$, but only that part in which we are interested, that is, $_{11}\{(bW_{y*y*})'bW_{y*y*}\}$. In this example, $(bW_{y*y*})'bW_{y*y*}$ is a 2x2 matrix, so that if the first row and first column are omitted, we are left with a scalar, which is obtained by squaring the second element of bW_{y*y*}. That is, $_{11}\{(bW_{y*y*})'bW_{y*y*}\} = (-0.6801)(-0.6801) = 0.4625$. The reason for this will be clear to the reader if he writes down the product matrix that would result from the multiplication of a column vector by a row vector, where each vector contains the same elements. The check on this operation is one of recomputation. If, however, there were more than two elements in y*, $_{11}\{(bW_{y*y*})'bW_{y*y*}\}$ would be a matrix.

We now compute

$D = 1 / \lambda$ (λ was obtained in section (9).)

$\dfrac{D}{bW_{y*y*}b'}$ ($bW_{y*y*}b'$ was obtained in section (17).)

$1 + D$

$C = (1 + D)\, bW_{y*y*}b'$

$C^* = C/N'$, where N' is the sample size minus the total number of variables in the equation. These computations are shown in section (19).

We now compute $_{11}G = \frac{D}{bW_{y*y*}b'} \times {}_{11}\{(bW_{y*y*})'bW_{y*y*}\}$. See sections (18), (19), and (20) of table 9. Sections (18) and (19) were obtained previously. (20) $_{11}G$ is obtained by multiplying $_{11}\{(bW_{y*y*})'bW_{y*y*}\}$, obtained in section (18), by $\frac{D}{bW_{y*y*}b'}$, computed in section (19).

We next compute $_{11}H = {}_{11}B - {}_{11}G$. See sections (20), (21), and (22) of table 9. Section (20) was obtained previously. (21) $_{11}B$ is obtained by eliminating the first row and first column of B_{y*y*}, obtained in section (8). (22) $_{11}H$ is obtained by subtracting (20) from (21). It should be noted that in this example sections (20), (21), and (22) are scalars. If there were more than two elements in y*, they would be matrices and the usual checks on matrix subtraction would be carried out.

We now compute $F_{bb} = {}_{11}H^{-1}$. See section (23) of table 9. (23) F_{bb} is obtained by taking the inverse of $_{11}H$, computed in section (22). In this example $_{11}H$ is a scalar and its inverse is the reciprocal of that number. If, however, $_{11}H$ were a matrix, other methods of matrix inversion would be used (see p. 26).

We next compute $F'_{bc} = {}_{01}P'F_{bb}$. See sections (23), (24), and (25) of table 9. Section (23) was obtained previously, (24) $_{01}P'$ is obtained by eliminating the first column from the matrix P' obtained in section (6). (25) F'_{bc} is obtained by multiplying $_{01}P'$ by F_{bb} . Note again that $_{01}P'$, F_{bb}, and F'_{bc} are scalars. If they were higher order matrices, matrix multiplication and the usual checks would be performed.

We now compute $U = {}_{01}P'F_{bc}$. See sections (24), (25), and (26) of table 9. Sections (24) and (25) were obtained previously. (26) Since $_{01}P'$ and F'_{bc} are both scalars, U is obtained by multiplying $_{01}P'$ by F'_{bc}. If such were not the case, matrix multiplication would be required, keeping in mind that the product involves the transpose of F'_{bc}. An efficient method to perform this operation is described on pp. 47-48.

We next compute $F_{cc} = U + M^{-1}_{z*z*}$. See sections (26), (27), and (28) of table 9. Section (26) was obtained previously. (27) M^{-1}_{z*z*} was computed in section (4). (28) F_{cc} is obtained by adding (26) and (27). If higher order matrices were involved, the usual checks on matrix addition or subtraction would be performed (see p. 37). The check here, as in the computations above where only scalars are involved, is one of recomputation.

We now compute C^*F_{bb}. See section (29) of table 9. (29) C^*F_{bb} is obtained by multiplying F_{bb}, obtained in section (23), by C*, computed in section (19). The diagonal elements of C^*F_{bb} are the variances (square of the standard error) of the b's, excluding b_1. The variance of b_1 equals zero by definition. In this example, C^*F_{bb} is a scalar. Therefore the variance of b_2 equals C^*F_{bb}. As in earlier examples, the coefficients and the variances need to be deadjusted to apply to the original data. This is discussed in a later section.

We next compute C^*F_{cc}. See section (30) of table 9. (30) C^*F_{cc} is obtained by multiplying F_{cc}, obtained in section (28), by C^*, computed in section (19). The diagonal elements of C^*F_{cc} are the variances of the c's. In this example, there is only one c, that is, c_1. Therefore, C^*F_{cc} is a scalar, and the variance of c_1 equals C^*F_{cc}.

When more than 2 y* or more than 1 z* are involved, only the diagonal elements of C^*F_{bb} and C^*F_{cc} need be computed.

We now proceed to estimate the variances of the coefficients of the just identified equation (1.1). The relevant computations are shown in table 10. Sections (14), (15), (16), and (17) are obtained in exactly the same way as for the overidentified equation (see table 10 and p. 44).

For a just identified equation, there is no section (18) comparable to that for an overidentified equation.

We next compute $C^* = \dfrac{bW_{y^*y^*}b'}{N'}$, where N' equals the number of observations minus the total number of variables in the equation. See section (19.2) of table 10. (19.2) C^* is obtained by dividing $bW_{y^*y^*}b'$, which was computed in section (17), by N'. For this example, $bW_{y^*y^*}b'$ is divided by $26 - 4 = 22$. Note that the subscripts on $W_{y^*y^*}$ have been omitted in this part of table 10.

Computations comparable to those of sections (20) to (22) of the overidentified equation, table 9, are not obtained for a just identified equation.

We now compute $F_{bb} = {}_{11}B^{-1}$. See section (23.2) of table 10. (23.2) F_{bb} is obtained by taking the inverse of ${}_{11}B$. As this computation was performed in section (9.2), section (9.2) is copied into section (23.2).

We next compute: $F'_{bc} = {}_{01}P'F_{bb}$. See sections (23.2), (24), and (25) of table 10. Section (23.2) was obtained previously. (24) ${}_{01}P'$ is obtained by eliminating the first column of the matrix P' obtained previously in section (6). This column, including Σ', is copied from section (6). That the sum of the elements of the copied column of ${}_{01}P'$ is identical with the item in the Σ' row is indicated by a check mark. (25) F'_{bc} is obtained by multiplying the elements of ${}_{01}P'$, including Σ', by F_{bb}. That the sum of the elements of the column of F'_{bc} is identical (except for possible rounding error) with the element in the Σ' row is indicated by a check mark. This checks the computation. It should be noted that F_{bb} in this example is a scalar. If there were more than two elements in y*, F_{bb} would be a matrix, and the usual matrix multiplication would be performed.

We now compute $U = {}_{01}P'F_{bc}$. See sections (24), (25), and (26). Sections (24) and (25) were obtained previously. (26) U is a matrix whose ijth element is equal to the product of the element in the ith row of ${}_{01}P'$, excluding Σ', by the element in the jth row of F'_{bc}, including Σ'. In this example, if we let u_{ij} be the elements of U:

$$u_{11} = (0.6323)\ (1.7738) = 1.1217$$

$$u_{12} = (0.6323)\ (0.9490) = 0.6001$$

$$u_{1\Sigma} = (0.6323)\ (2.7228) = 1.7218$$

$$u_{22} = (0.3383)\ (0.9490) = 0.3210$$

$$u_{2\Sigma} = (0.3383)\ (2.7228) = 0.9212$$

That the sums across the rows of U are identical, except for possible rounding errors, with the elements in the respective rows of the Σ column of U is indicated by check marks. Note that row-by-row multiplication is used here. If ${}_{01}P'$ and F'_{bc} were higher order matrices instead of vectors, as would be the case if y^* had more than two elements, U would be a matrix whose ijth element would be the cumulative product of the elements in the ith row of ${}_{01}P'$, excluding the row Σ', by the elements in the jth row of F'_{bc}, including Σ' (see p.101).

We now compute $F_{cc} = U + M^{-1}_{z^*z^*}$. See sections (26), (27), and (28). Section (26) was obtained previously. (27) $M^{-1}_{z^*z^*}$ was obtained in section (4). It can be copied into section (27); the usual check on copying matrices is performed (see p. 37). If preferred, section (27) can be omitted. (28) F_{cc} is obtained by adding (26) and (27), including their respective Σ's. That the sums across the rows of F_{cc} are identical, (except for errors that result from carrying only 4 decimals) with the elements in the respective rows of the Σ column of F_{cc} is indicated by check marks.

We next compute C^*F_{bb}. See section (29). (29) C^*F_{bb} is obtained by multiplying F_{bb}, obtained in section (23.2), by C^*, computed in section (19.2). C^*F_{bb} equals the variance of b_2.

We now compute C^*F_{cc}. See section (30). (30) C^*F_{cc} is obtained by multiplying F_{cc}, obtained in section (28), by C^*, computed in section (19.2). As explained before, the diagonal elements of C^*F_{cc} are the variances of the c's. The element in the upper left-hand corner is the variance of c_1; the element in the lower right-hand corner is the variance of c_2. Standard errors of the coefficients are obtained by taking the square root of the variances of the respective coefficients. If desired, only the diagonal elements of this matrix need be computed.

We now return to the equations in their original form, that is, equations (1) and (2). The computations involved are shown at the bottom of tables 9 and 10. The following explanation is in terms of the overidentified equation, table 9.

Column (1): List the variables in the order that they appear in equation (2).

Column (2): Enter the estimates of the coefficients in equation (2). This is done by referring to the relationship between the coefficients in the original equation (2) and the rewritten form for computational purposes, equation (2.1), given on p. 31, and to the b' in section (12) and the c' in section (13). For example, the coefficient on Y_2 is 1, since the coefficient of the first variable is always 1. $b_{21} = -b_2^2$. Since $b_2{}^2$, the second element in b', equals -0.8555, b_{21} = 0.8555. Similarly, $c_{23} = -c_1^2$. Since c_1, the element in c', equals 0.2409, c_{23} = -0.2409.

Column (3): Enter the estimates of the standard errors of the coefficients in equation (2). There is no standard error for the first coefficient. The standard error of b_{21} equals the standard error of b_2, which is obtained by taking the square root of C^*F_{bb}, section (29). If there were more than two elements in y* and hence more than two b's, their standard errors would be obtained by taking the square roots of the diagonal elements of C^*F_{bb}, which would then be a matrix. The standard error of c_{23} equals the standard error of c_1, which is obtained by taking the square root of C^*F_{cc}, section (30). If more than one element were in z*, as in equation (1.1), the standard errors of the c's would be obtained by taking the square roots of the diagonal elements of C^*F_{cc} (see p. 62).

Column (4): Copy the k_i for the respective variables from table 6.

Column (5): Compute k_i', obtained by dividing k_i by the value of k for the first variable in column (1). That is, k_i' for Y_1 and Z_3 is obtained by dividing their respective k_i by 0.001, the value of k for Y_2.

Column (6): Deadjusted coefficients are obtained by multiplying the elements of column (2) by those of column (5).

Column (7): Deadjusted standard errors are obtained by multiplying the elements of column (3) by those of column (5).

Column (8): Enter the means of the variables, which are obtained from table 5.

Column (9): Computations in this column are used in obtaining the constant for the equation. Multiply the elements of column (8) by those of column (6) and compute their sum. This can be cumulated directly in the machine. The constant in the equation, a, is obtained by subtracting this sum from the mean of Y_2, the first element in column (8). Hence a = 147.7307 - (-2.6520) = 150.3828. This result can be recorded directly as the first term to the right of the equality sign in the final regression equation shown in the last row of this section. The figures within the parentheses are the deadjusted standard errors of the coefficients.

The check in this section is one of recomputation.

Deadjusting the coefficients and their standard errors, and the writing of equation (1), are accomplished in like manner in table 10.

Modifications with Specified Numbers of Endogenous and Predetermined Variables

In the last section we described computations involved in applying the limited information approach to a 2-equation model of a given type. Here we discuss modifications in the procedure required when the equations involved are of a different type. These modifications result from the fact that we frequently work with equations where the number of elements in y^*, in z^*, or the degree of identification differ from those of the equations in the lumber model. The modifications may appear in the form of alternate steps--for example, the case of an overidentified equation where the number of elements in y^* is three or more--or auxiliary steps--for example, a variation of the Doolittle method to invert $M_{z^*z^*}$ where the number of elements in z^* is three or more. In addition, the form of many of the matrices shown in tables 9 and 10 may be changed due to an increased number of variables. For example, a scalar may become a vector, or a vector a larger matrix. Furthermore, the increased size of the matrices sometimes necessitates the introduction of additional checks on matrix operations.

Table 11 summarizes some of these modifications; others are explained in the text. The section numbers in table 11 are the same as those in tables 9 and 10. The section numbers of table 9 are taken as a standard. They represent the necessary steps for an overidentified equation with two elements in y^* and one element in z^*. Additional section numbers are indicated by the following scheme: (1) A number with a digit to the right of the decimal indicates an alternative method necessitated by the identification of the equation. For example, (9.1) and (9.2) refer, respectively, to the alternates to section (9) of table 9 for (a) an overidentified equation whose elements in y^* are three or more, and (b) a just identified equation, regardless of the number of elements in y^* or z^*. A section number followed by ".1" refers to an overidentified equation; one followed by ".2", to a just identified one. (2) A number followed by a letter indicates an auxiliary step necessary to produce the matrix for the section indicated by the number. For example, (4c) indicates a variation of the Doolittle method to invert $M_{z^*z^*}$ required for section (4). "c" always refers to a variation of the Doolittle method to invert a matrix whose order is three or more. (3) A number to the right of the decimal <u>and</u> a letter combine the two previous attributes. For example, (9.2c) refers to a variation of the Doolittle method to invert the matrix required for section (9.2), which itself is a variation applicable to just identified equations.

We first discuss table 11, section by section, (1) to note changes in the size of the matrices with an increasing number of variables, (2) to introduce new checks on the matrix operations and (3) to explain new or additional steps.

In the following explanation, the letter <u>g</u> denotes the number of elements in y^*, and <u>h</u>, the number of elements in z^*. A matrix of order <u>g</u> refers to a matrix having <u>g</u> rows and <u>g</u> columns. Furthermore, when we state that a matrix has <u>g</u> rows (columns), we know that the first row (column) corresponds to the

Table 11.- Alternative steps involved in computations for equations with specified characteristics

Section	
(1) $M_{y^*z}M_{zz}^{-1}M_{zy^*}$	Same procedure for all
(2) $M_{y^*y^*}$	
(3) $W_{y^*y^*}$	

	h = 1 1/	h = 2 1/	h ≥ 3 1/
(4) $M_{z^*z^*}^{-1}$	(4), table 9	(4a), (4b), table 10	(4c), p. 26

(5) $M_{z^*y^*}$	Same procedure for all
(6) P'	
(7) $M_{y^*z^*}P'$	
(8) $B_{y^*y^*}$	

	Overidentified equation		Just identified equation		
	g = 2 2/	g ≥ 3 2/	g = 2 2/	g = 3 2/	g ≥ 4 2/
(9)	(9), table 9	(9.1), pp. 53-54	(9.2), table 10	(9.2a), (9.2b), p. 57	(9.2c), p. 58
(10)	(10), do.	(10.1), pp. 53-55	(10.2), table 10 and pp. 57-58		
(11)	(11), do.	(11.1), pp. 55-56	(11.2), Do.		
(12)	(12), do.	(12.1), pp. 56-57	(12.2), Do.		

(13) c'	Same procedure for all
(14) b	
(15) $W_{y^*y^*}$	
(16) $bW_{y^*y^*}$	
(17) $bW_{y^*y^*}b'$	

	Overidentified equation	Just identified equation
(18)	(18), table 9 and p. 59	No comparable section
(19)	(19), do.	(19.2), table 10 and p. 60
(20)	(20), do.	No comparable section
(21)	(21), do.	Do.
(22)	(22), do.	Do.

	Overidentified equation			Just identified equation		
	g = 2 2/	g = 3 2/	g ≥ 4 2/	g = 2 2/	g = 3 2/	g ≥ 4 2/
(23)	(23), table 9	(23a), (23b), pp. 59-60	(23c), p. 60	(23.2), copy results from (9.2)		

(24) $_{01}P'$	Same procedure for all
(25) F'_{bc}	
(26) U	
(27) $M_{z^*z^*}^{-1}$	
(28) F_{cc}	
(29) C^*F_{bb}	
(30) C^*F_{cc}	

2/ g denotes the number of elements in y*.

first element of y^*, 16/ and is so designated; the second row (column) corresponds to the second element of y^*, and so forth; so that, in general, the ith row (column) corresponds to the ith element of y^*. A similar statement can be made for a matrix with h rows or columns by substituting z^* for y^*.

Section (1): $M_{y*z}M_{zz}^{-1}M_{zy*}$.--The computation of this matrix is explained on p. 31. There are g columns in addition to a Σ column composed of row sums. It is a symmetrical matrix of order g. As with all symmetrical matrices, the terms below the main diagonal need not be computed; however, they must be included in obtaining the row sums. The computed matrix, including Σ, is copied into section (1) and the check on copying is performed as explained on p. 37.

Section (2): M_{y*y*} .--The formation of this matrix is explained on p. 37. It is a symmetrical matrix of order g, with an additional Σ column.

Section (3): W_{y*y*} .--The computation of this matrix and its check is explained on p. 37. It is a symmetrical matrix of order g, with an additional Σ column.

Section (4): M_{z*z*}^{-1} .--The computation of this matrix is determined by the number of elements in z^*, h:

If h = 1, see section (4), table 9, and p. 37.
If h = 2, see sections (4a) and (4b) of table 10, and p. 40.
If h $\geq$ 3, use a variation of the Doolittle method to invert M_{z*z*}. This constitutes section (4c). For an explanation of the computations involved, as well as the check on this computation, see p. 26. This computation is best carried out on a separate worksheet and the result, M_{z*z*}^{-1}, including Σ, the column of row sums, and the missing elements, entered into section (4). At times use can be made of part of the computations necessary to compute M_{zz}^{-1}. (See p. 75.)

M_{z*z*}^{-1} is a symmetrical matrix of order h, with an additional Σ column.

Section (5): M_{z*y*} .--The formation of this nonsymmetrical matrix is explained on p. 37. It has h rows and g columns, with an additional Σ column.

Section (6): P' .--This nonsymmetrical matrix is obtained by multiplying M_{z*z*}^{-1} by M_{z*y*}. Scalar multiplication and its check are explained on p. 40. Matrix multiplication and its check are explained on p. 24. Like M_{z*y*}, P' has h rows and g columns, in addition to a Σ column. A row, Σ', composed of column sums is obtained if there are two or more rows in P' and is used in later computations. (See sections (13), (24) and (25).) All elements of this matrix must be computed.

16/ See p. 32.

Section (7):$M_{y*z*}P'$.--This computation is explained on p. 37, and is summarized here. To obtain the ijth element of $M_{y*z*}P'$, cumulate the products of the terms of the ith column of M_{z*y*}, excluding Σ, with the jth column of P', including Σ but excluding the row Σ'. That the sums across the rows of $M_{y*z*}P'$ are identical (except for possible rounding errors) with the items in the respective rows of the Σ column of $M_{y*z*}P'$ is indicated by check marks. This provides the check on the computation. $M_{y*z*}P'$ is a symmetrical matrix of order g, with an additional Σ column. Note that column-by-column multiplication is used in this step.

Section (8):B_{y*y*} .--The computation of this matrix is explained on p. 38. It is a symmetrical matrix of order g, with an additional Σ column.

Up to this point the steps are exactly the same regardless of the degree of identification of the equation, the only variation arising from the number of elements in z*. However, for sections (9) - (12), as indicated in table 11, there are alternative methods according to both the degree of identification and the number of elements in y*.

Sections (9) - (12):For Overidentified Equations.--(1) If $g = 2$, the procedure is exactly that explained on p. 38 and shown in sections (9), (10), (11), and (12) of table 9. (2) If $g \geq 3$, the procedure of sections (9.1), (10.1), (11.1), and (12.1) should be used. The computations and explanations which follow are in terms of equation (2.1), where $g = 2$, as the method is applicable for any number of elements in y* greater than one. 17/ However, if $g = 2$, the method shown in sections (9), (10), (11), and (12) of table 9 is computationally more efficient and hence should be used. The present example is given only as a simplified case for illustrative purposes.

In sections (9.1), and (10.1) of table 12, compute A', the transpose of $A = B^{-1}_{y*y*}W_{y*y*}$, using another variation of the Doolittle method. The forward solution is carried out in section (9.1) and a back solution in section (10.1). Explanation is as follows: In the upper section of the forward Doolittle solution, write B_{y*y*} and W_{y*y*}, obtained from sections (8) and (3), respectively. Note that W_{y*y*} is written out in full. Compute also a Σ column, composed of row sums, including those terms in B which are omitted because of symmetry. The encircled numbers are used for illustrative purposes in the outline of the computation of A' and its check. Carry out the forward Doolittle solution in rows 1 - 2" as explained on p. 9. Note that only one Σ column is used to check the computation of the entire row. In section (10.1), the computation of A' is in the form of a back solution. This involves a square matrix of order g. The outline showing the computation of the a_{ij}, the elements of A', is self-explanatory. The subscripts of the a_{ij} refer to the rows and columns of A'; they have no reference to the subscripts of the y's. The

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17/ When there is only one element in y*, we have an ordinary multiple regression equation which can be handled by the least squares method explained in detail in the first part of this handbook, or by the methods described on pp. 70-75.

Table 12.- Computations involved in obtaining sections (9.1) and (10.1) for an overidentified equation for which g=2 1/

Section (9.1) - Forward solution

Row	B_{y*y*} y_2	B_{y*y*} y_1	W_{y*y*} y_2	W_{y*y*} y_1	Σ
y_2	0.6688	0.7689	0.3183	0.2550	2.0112
y_1		.9858	.2550	1.0930	3.1028
(1)	0.6688	0.7689	0.3183	0.2550	2.0112
(1")	1.	1.1496 ①	.4759 ②	.3812 ③	3.0069 ④ ✓
(2)		.9858	.2550	1.0930	3.1028
(1)(-1.1496)		-.8840	-.3660	-.2931	-2.3122
(2')		.1017	-.1110	.7998	.7905 ✓
(2")		1.	-1.0912 ⑤	7.8617 ⑥	7.7704 ⑦ ✓

Section (10.1) - Back solution

Computation of A'

a'_{11}	a'_{12}
0.4759	-1.0912
1.2546	
1.7305	
a'_{21}	a'_{22}
.3812	7.8617
-9.0386	
-8.6573	
$a'_{\Sigma 1}$	$a'_{\Sigma 2}$
3.0069	7.7704 ✓
-8.9336	
-5.9267 ✓	

Outline

$$\frac{② \quad a'_{12}(-①)}{a'_{11}} \qquad ⑤ = a'_{12}$$

$$\frac{③ \quad a'_{22}(-①)}{a'_{21}} \qquad ⑥ = a'_{22}$$

$$\frac{④ \quad a'_{\Sigma 2}(-①)}{a'_{\Sigma 1}} \qquad ⑦ = a'_{\Sigma 2}$$

Check:

$a'_{12} + a'_{22} + 1 = a'_{\Sigma 2}$

$a'_{11} + a'_{21} + 1 = a'_{\Sigma 1}$

1/ These computations were performed with 9 decimal places, of which only 4 appear in the table. Therefore some of the computations may appear to be slightly in error. This method is used only when g≥3. This simplified example is shown for purposes of illustration.

check on the computation of A' also is self-explanatory. These computations can be easily extended for any number of elements in y*. (See appendix, p. 92.)

In section (11.1) we estimate λ by an iterative method. 18/ Computations are shown in table 13. The explanation is as follows: Write A', computed in section (10.1). Next to it write the vector $q^{(0)}$, which is a column of 1's. $q^{(0)}$ has as many 1's as the number of rows in A'. Beneath it write $Q^{(1)}$ whose elements, $Q_i^{(1)}$, are obtained in the following manner: $Q_1^{(1)}$ equals the sum of the elements of the first column of A', $Q_2^{(1)}$ equals the sum of the elements of the second column of A'. In general, therefore, the ith element of $Q^{(1)}$ is obtained by summing the ith column of A'. These sums (plus 1) were computed in the check on the computations of A'. For our example,

$$Q_1^{(1)} = (1.7305) + (-8.6573) = -6.9267$$

$$Q_2^{(1)} = (-1.0912) + (7.8617) = 6.7704$$

Table 13.- Computations involved in obtaining section (11.1) for an overidentified equation 1/

A'	Successive iterations						
	$q^{(0)}$	$q^{(1)}$	$q^{(2)}$	$q^{(3)}$	$q^{(4)}$	$q^{(5)}$	$q^{(6)}$
1.7305 -1.0912	1.	1.	1.	1.	1.	1.	1.
-8.6573 7.8617	1.	-.9774	-.8609	-.8558	-.8555	-.8555	-.8555
	$Q^{(1)}$	$Q^{(2)}$	$Q^{(3)}$	$Q^{(4)}$	$Q^{(5)}$	$Q^{(6)}$	$Q^{(7)}$
	-6.9267	10.1926	9.1843	9.1395	9.1373	9.1372	9.1372
	6.7704	-8.7756	-7.8600	-7.8193	-7.8173	-7.8172	-7.8172

1/ These computations were performed with 9 decimal places, of which only 4 appear in the table. Because of this some of the computations may appear to be slightly in error.

$q_i^{(1)}$, the elements of $q^{(1)}$, are obtained by dividing each of the $Q_i^{(1)}$ by the element of $Q^{(1)}$ having the largest absolute value. In the present example, the largest element of $Q^{(1)}$ is -6.9267. $q_1^{(1)}$, therefore, equals -6.9267/-6.9267

18/ This method differs from that discussed by Chernoff and Divinsky (2, p. 244) and Klein (13, p. 183).

= 1; $q_2^{(1)}$ = 6.7704/-6.9267 = -0.9774. $Q_i^{(2)}$, the elements of $Q^{(2)}$, are obtained in the following manner: $Q_1^{(2)}$ equals the sum of the products of the elements of the first column of A' with the elements of $q^{(1)}$; $Q_2^{(2)}$ equals the sum of the products of the elements of the second column of A' with the elements of $q^{(1)}$. In general, therefore, the ith element of $Q^{(2)}$ equals the sum of the products of the ith column of A' with the elements of $q^{(1)}$. In our example,

$$Q_1^{(2)} = (1.7305)\ (1) + (-8.6573)\ (-0.9774) = 10.1926$$

$$Q_2^{(2)} = (-1.0912)\ (1) + (7.8617)\ (-0.9774) = -8.7756$$

$q_i^{(2)}$, the elements of $q^{(2)}$, are obtained by dividing each of the $Q_i^{(2)}$ by the element of $Q^{(2)}$ having the largest absolute value. In our example, therefore, $q_1^{(2)}$ = 10.1926/10.1926 = 1; $q_2^{(2)}$ = -8.7756/10.1926 = -0.8609. $Q_i^{(3)}$, the elements of $Q^{(3)}$, are obtained in the following manner: $Q_1^{(3)}$ equals the sum of the products of the elements of the first column of A' with the elements of $q^{(2)}$; $Q_2^{(3)}$ equals the sum of the products of the elements of the second column of A' with the elements of $q^{(2)}$; and in general, $Q_i^{(3)}$ equals the sum of the products of the ith column of A' with the elements of $q^{(2)}$. In our example,

$$Q_1^{(3)} = (1.7305)\ (1) + (-8.6573)\ (-0.8609) = 9.1843$$

$$Q_2^{(3)} = (-1.0912)\ (1) + (7.8617)\ (-0.8609) = -7.8600$$

Continue in this manner, that is, $q_i^{(k)} = Q_i^{(k)} / Q_j^{(k)}$, where $Q_j^{(k)}$ is the element in $Q^{(k)}$ having the largest absolute value; and $Q_i^{(k+1)}$ equals the sum of the products of the ith column of A' by the $q_i^{(k)}$, until the elements in the successive Q vectors, that is, $Q_i^{(k)}$ and $Q_i^{(k+1)}$, agree to the number of required decimal places. Six iterations were required for accuracy to 4 decimals as shown in table 13; nine iterations were needed for the required agreement of nine decimal places when the full computations were carried out. 19/ The largest element of the final computed Q vector, that is, $Q_j^{(6)}$ in table 13, is the estimate of λ. The only check on the computations needed is that of recomputing the final Q and q vectors; all other checks are unnecessary owing to the nature of the iterative process. The reader will note that, except for a rounding error due to dropping decimals, the estimate for λ obtained in table 13 is the same as that shown in section (9), table 9.

In section (12.1) we estimate b', the vector of the coefficients of the endogenous variables. b' is obtained from the final q vector, computed in section (11.1), in the following manner: If the first element of the final q vector equals 1, b' is this final q vector; if the first element of the final q vector does not equal 1, b' is obtained by dividing each element of the

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19/ Some authors recommend raising the matrix to the eighth power before beginning the iterations. When this was tried on this example, the number of necessary iterations was reduced to three. But the additional work involved in raising the matrix to a higher power, which requires use of a "floating" decimal point on desk calculators, outweighs the advantages.

final q vector by its first element. For example, if the elements of the final q vector were 0.2 and 0.4, the elements of b' would be 1.0 and 2.0 respectively. In our case, the first element of the final q vector, $q^{(6)}$, equals 1 and the second element equals -0.8555. These results are the same as those shown in section (12) of table 9. If there are more than two elements in b', compute the sum of all the elements in the vector and enter that sum in a Σ' row. This sum is used later.

Sections (9)-(12):For Just Identified Equations.--(1) If g=2, the procedure is that explained on p. 43 and shown in sections (9.2), (10.2), (11.2), and (12.2) of table 10.

(2) If g=3, sections (9.2a) and (9.2b) are used to invert the matrix ${}_{11}B$, which is required for section (9.2). ${}_{11}B$ is a symmetrical matrix obtained by omitting the first row and first column of B_{y*y*}, computed in section (8). A column, Σ, is obtained for ${}_{11}B$ by subtracting the first element in each row other than the first of B_{y*y*} from Σ of B_{y*y*}. That the sum across the rows of ${}_{11}B$ is identical with the figure appearing in the Σ column of ${}_{11}B$ is indicated by a check mark. When y* has 3 elements, ${}_{11}B$ is a 2x2 matrix. The method used to invert a matrix of order 2, including the check, is explained on p. 26, and shown in sections (4a) and (4b) of table 10. Sections (9.2a) and (9.2b) are analagous to sections (4a) and (4b), respectively, M_{z*z*} being replaced by ${}_{11}B$. ${}_{11}B^{-1}$, including the missing elements, is then entered in section (9.2). A row, Σ', consisting of column sums, should be computed; this enters into later computations.

In section (10.2) write B_1, the first column of B_{y*y*} with its first element omitted. Before obtaining B_1, the missing elements in B_{y*y*} must be filled in, at least mentally. When g=3, B_1 is a column vector consisting of two rows. An additional row, Σ', is obtained by subtracting the element in the first row and first column of B_{y*y*} from the item in the first row and Σ column of B_{y*y*}. That the sum of the items in B_1 is identical with the corresponding value in the Σ' row is indicated by a check mark. This provides a check on the copying. Σ', however, is not used in later computations.

In section (11.2) compute ${}_1b'$, which is b', the column vector of the coefficients of the endogenous variables, with its first element omitted. The ith element of ${}_1b'$ is obtained by cumulating the product of the elements of the ith row of $-[{}_{11}B]^{-1}$, excluding Σ but including Σ', with the elements of B_1, excluding Σ'. That the sum of the elements of ${}_1b'$ is identical (except for possible rounding error) with the element in the Σ' row of ${}_1b'$ is indicated by a check mark. Computationwise, it is easier to neglect the minus sign prefixed to ${}_{11}B^{-1}$. Instead, multiply ${}_{11}B^{-1}$, including Σ', by B_1, excluding Σ'; change the sign of the result, ${}_1b'$, including Σ'; and perform the check in the manner indicated previously.

In section (12.2) b' is obtained from ${}_1b'$ by inserting the omitted element, b_1, which equals 1 by definition. b' is a column vector having 3 rows when y* has 3 elements. An additional row, Σ', is obtained by adding 1 to the element in the Σ' row of ${}_1b'$. That the sum of the elements in b' is identical with the element in the Σ' row of b' is indicated by a check mark. This checks the copying.

(3) If $g \geq 4$, a variation of the Doolittle method is used to invert ${}_{11}B$, which is required for section (9.2), and this is considered as section (9.2c). For an explanation of the computations involved, as well as the check on this operation, see p. 26. This computation is best carried out on a separate worksheet and the result, ${}_{11}B^{-1}$, including the missing elements, entered into section (9.2). An additional row, Σ', consisting of column sums should be computed. ${}_{11}B$ and its inverse are symmetrical matrices of order g-1. In sections (10.2), (11.2) and (12.2), the computations are the same as those just given for the case where g=3. B_1 (section 10.2) and ${}_1b'$ (section 11.2) are column vectors with g-1 rows, in addition to a Σ' row. b' (section 12.2) is a column vector with g rows, in addition to a Σ' row.

With section (13), see table 11, the computations are applicable regardless of the degree of identification of the equation.

Section (13):c'.--This is obtained by multiplying -P', obtained in section (6), by b', obtained in section (12) for an overidentified equation where g=2, section (12.1) for an overidentified equation where $g \geq 3$, or section (12.2) for a just identified equation. The ith element of c' is obtained by cumulating the product of the elements of the ith row of -P', excluding Σ but including Σ', with the elements of b', excluding Σ'. That the sum of the elements of c' is identical (except for possible rounding error) with the element in the Σ' row of c' is indicated by a check mark. c' is a column vector with h rows in addition to a Σ' row. Computationwise, it is easier to neglect the minus sign prefixed to P'. Instead, multiply P', including Σ' and excluding Σ, by b' excluding Σ'; change the sign of the resulting c', including Σ', and perform the check. This procedure is used when P' has a Σ' row, that is, when $h \geq 2$. If however, h = 1, as in table 9, c' is a scalar and the check is one of recomputation.

Beginning with section (14) computations are started for the standard errors of the coefficients just obtained.

Section (14):b.--This row vector with g columns is obtained by transposing the column vector b', obtained in section (12) for an overidentified equation where g=2; section (12.1) for an overidentified equation where $g \geq 3$; or section (12.2) for just identified equations regardless of the number of elements in y*. When $g \geq 3$, the element in the Σ' row of b' becomes the element in the Σ column of b. That the sum of the elements of the transposed vector is identical with the element in the Σ column of b is indicated by a check mark. This checks the transposition.

Section (15):$W_{y^*y^*}$.--This matrix was computed in section (3). (See p. 37.) It is copied into section (15). The missing elements are included in the copied version. The usual check on copying should be used.

Section (16):$bW_{y^*y^*}$.--This computation is explained on page 45. $bW_{y^*y^*}$ is a row vector having g columns in addition to a Σ column.

Section (17):$bW_{y^*y^*}b'$.--This scalar is obtained by cumulating the products of the ith element of $bW_{y^*y^*}$, excluding Σ (obtained in section (16)), with the ith element of b, excluding Σ (obtained in section (14)). Note that $bW_{y^*y^*}$

and b are always row vectors and that $bW_{y^*y^*}b'$ is always a scalar. As with all scalars, the check is one of recomputation.

Sections (18) - (23):For Overidentified Equations Only.--In section (18) we obtain $_{11}\{(bW_{y^*y^*})'bW_{y^*y^*}\}$ by omitting the first row and first column of $(bW_{y^*y^*})'bW_{y^*y^*}$, a symmetrical matrix of order g. Its ijth element is obtained by multiplying the ith element of $bW_{y^*y^*}$ with the jth element of $bW_{y^*y^*}$. We do not, however, compute the entire matrix, $(bW_{y^*y^*})'bW_{y^*y^*}$, but only that part in which we are interested, that is, all rows and columns other than the first. For example, if y* had four elements, and we denote the elements of $(bW_{y^*y^*})'bW_{y^*y^*}$ as b_{ij}, we would not compute b_{11}, b_{12}, b_{13}, or b_{14}. We would not compute b_{21}, b_{31}, or b_{41} in any event, as it is symmetrical. This computation and its check are explained on page 45 for the case where g = 2. Note that the Σ column of $bW_{y^*y^*}$ is not used in these computations. When $g>2$, an additional sum, $_1\Sigma$, is computed for $bW_{y^*y^*}$. Since Σ is the sum of all the elements of $bW_{y^*y^*}$, $_1\Sigma$ is equal to Σ minus the first element of $bW_{y^*y^*}$. The element in the ith row and Σ column of $_{11}\{(bW_{y^*y^*})'bW_{y^*y^*}\}$ is obtained by multiplying the ith element of $bW_{y^*y^*}$ by $_1\Sigma$. That the sums across the rows of $_{11}\{(bW_{y^*y^*})'bW_{y^*y^*}\}$ are identical (except for possible rounding errors) with the items in the Σ column of the respective rows is indicated by check marks. This checks the computation.

See table 9 and p. 45 for computations involved in section (19). Except for λ, which is obtained in section (9) when g = 2 and in section (11.1) (see p. 55) when $g \geq 3$, the computations are the same regardless of the number of elements in y*.

In section (20), we obtain $_{11}G$ by multiplying $_{11}\{(bW_{y^*y^*})'bW_{y^*y^*}\}$ by $\frac{D}{bW_{y^*y^*}b'}$. $_{11}\{(bW_{y^*y^*})'bW_{y^*y^*}\}$ is the symmetrical matrix of order g-1 obtained in section (18). $\frac{D}{bW_{y^*y^*}b'}$ is a scalar obtained in section (19). For an explanation of scalar multiplication and its check, see p. 40. $_{11}G$ is a symmetrical matrix of order g-1 with an additional Σ column (when $g \geq 3$).

In section (21) we obtain $_{11}B$, a symmetrical matrix, by omitting the first row and first column of $B_{y^*y^*}$, computed in section (8). A column, Σ, is obtained for $_{11}B$ (when $g \geq 3$) by subtracting the first element in each row, other than the first, from Σ of $B_{y^*y^*}$. That the sums across the rows of $_{11}B$ are identical with the elements in the Σ column of the respective rows of $_{11}B$ is indicated by check marks.

In section (22) we obtain $_{11}H$, a symmetrical matrix, by subtracting $_{11}G$, computed in section (20), from $_{11}B$, computed in section (21). Like $_{11}G$ and $_{11}B$, $_{11}H$ is of order g-1 and has an additional Σ column (when $g \geq 3$).

In section (23) we obtain F_{bb}. This matrix is the inverse of $_{11}H$, computed in section (22). If g = 2, $_{11}H$ is a scalar, and its inverse is obtained by taking the reciprocal of that number. (See p. 26 and table 9.) If g = 3, sections (23a) and (23b) are used to invert the 2x2 matrix, $_{11}H$, required for section (23). The method used to invert a 2x2 matrix and its check is explained on p. 26, and shown in sections (4a) and (4b) of table 10. Sections (23a) and (23b) are analogous to sections (4a) and (4b), respectively, when M_{z*z*} is replaced by $_{11}H$. $_{11}H^{-1}$ is then entered into section (23). If $g \geq 4$, a variation of the Doolittle method is used to invert $_{11}H$, which is required for section (23). This is done in section (23c). For an explanation of the computations involved and the check, see p. 26. This computation is best carried out on a separate worksheet, and the result, $_{11}H^{-1}$ and, Σ, the column of row sums, entered into section (23). Note that $_{11}H^{-1}$, or F_{bb}, is a symmetrical matrix of order g-1, when $g \geq 3$. The missing elements should be shown in section (23).

Sections (19.2) and (23.2):For Just Identified Equations Only.--For a just identified equation there is no section (18) comparable to that for an overidentified equation.

In section (19.2) we compute $C^* = bW_{y*y*}b'/N'$. N' equals the number of observations minus the total number of variables in the equation. $bW_{y*y*}b'$ is the scalar obtained in section (17).

Computations comparable to those of sections (20) - (22) for an overidentified equation are not obtained for a just identified equation.

In section (23.2) we obtain F_{bb} by taking the inverse of $_{11}B$, that is, the matrix obtained by omitting the first row and first column of B_{y*y*}, obtained in section (8). This computation was performed in section (9.2). We therefore copy section (9.2), including the Σ column which is obtained when g = 3 or $g \geq 4$, into section (23.2). Any missing elements should be included in the copied version.

With section (24) (see table 11) the computations are applicable regardless of the degree of identification of the equation.

Section (24):$_{01}P'$.--This nonsymmetrical matrix is obtained by eliminating the first column of P', computed previously in section (6), and copying the remaining columns, except the Σ column, into section (24). When $h \geq 2$, the elements of the Σ' row of P' are also copied into section (24). That the sums of the elements in the copied columns of $_{01}P'$ are identical with the elements in the respective columns of the Σ' row is indicated by check marks. This checks the copying. $_{01}P'$ has h rows, in addition to a Σ' row (when $h \geq 2$), and g-1 columns.

Section (25):F'_{bc}.--This matrix is obtained by multiplying $_{01}P'$, including Σ', by F_{bb}, excluding Σ. This is a nonsymmetrical matrix, so all elements must be computed. That the sums of the elements of the columns of F'_{bc} are identical (except for possible rounding errors) with the elements in the respective columns of the Σ' row of F'_{bc}, is indicated by check marks. F'_{bc} has h rows, in addition to Σ', and g-1 columns. This procedure holds only when $_{01}P'$ has a Σ' row, that is, when $h \geq 2$. If, however, h = 1 and g = 2, as in table 9, $_{01}P'$, F_{bb} and their product F'_{bc}, are scalars and the check is one of recomputation. On the other hand, if h = 1 and $g \geq 3$, $_{01}P'$ is a row vector and F_{bb}, a symmetrical matrix of order g - 1. F'_{bc} is obtained by multiplying ${}_{01}P'$ by F_{bb}, including Σ. That the sum of the elements of the resulting row vector, F'_{bc}, is identical (except for possible rounding error) with the elements in the Σ column of F'_{bc} is indicated by check marks. F'_{bc} is then a row vector with $g - 1$ columns in addition to a Σ column.

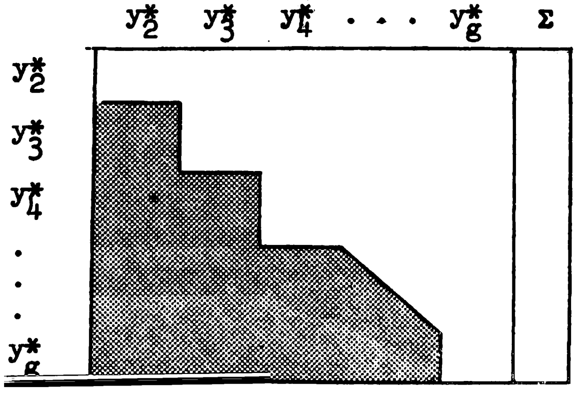
Many analysts will have an interest only in the standard error (the square root of the variance) of each coefficient. They will not wish to have the covariance terms computed. If only the standard errors (or variances) of the coefficients are desired, only the diagonal elements of each matrix need be computed starting with U in section (26).

Section (26): U.--The ijth element of this matrix is obtained by summing the products of the elements of the ith row of ${}_{01}P'$, excluding Σ', with the elements of the jth row of F'_{bc}, including Σ'. Note that row-by-row multiplication is used here. That the sums across the rows of U are identical (except for possible rounding errors) with the elements in the respective rows of the Σ column of U is indicated by check marks. U is a symmetrical matrix of order h with an additional Σ column. This procedure holds only when $h \geq 2$. If, however, $h = 1$ and $g = 2$, as in table 9, ${}_{01}P'$, F'_{bc} and their product U are scalars, and the check is one of recomputation. On the other hand, if $h = 1$ and $g \geq 3$, U is the scalar obtained by cumulating the products of the ith element of the row vector ${}_{01}P'$ with the ith element of the row vector F'_{bc}, excluding Σ. The check is one of recomputation.

Section (27): M^{-1}_{z*z*}.--This matrix, including the Σ column, which was obtained in section (4), is copied into section (27). If the covariances are not desired, only the diagonal elements need be copied. The usual check on the copying of matrices should be performed.

Section (28): F_{cc}.--This matrix is obtained by adding U and M^{-1}_{z*z*}, as explained on p. 46. It is a symmetrical matrix of order h, with an additional Σ column.

Section (29): C^*F_{bb}.--This matrix is obtained by multiplying F_{bb}, computed in sections (23) and (23.2), by C^*, computed in sections (19) and (19.2), for overidentified and just identified equations, respectively. Since C^* is a scalar, each element of F_{bb}, including those in the Σ column, is multiplied by C^*. That the sums across the rows of C^*F_{bb} are identical (except for possible rounding errors) with the elements in the respective rows of the Σ column of C^*F_{bb} is indicated by check marks. Like F_{bb}, C^*F_{bb} is a symmetrical matrix of order g-1, and will be of the form



The variance of b_i, the coefficient on y_i^*, (the ith element in the y* vector) is the element in the y_i^* row and y_i^* column of C^*F_{bb}. Note there is no row or column corresponding to y_1^*, since the variance of b_1 equals zero. The nondiagonal elements of C^*F_{bb} are the covariance terms for the coefficients on the endogenous variables, the b's. For example, the element in the y_3^* row and y_4^* column of C^*F_{bb} is the covariance b_3b_4, where b_3 and b_4 are the coefficients on y_3^* and y_4^*, respectively.

If only the variances are desired, only the diagonal elements of C^*F_{bb} need be computed.

Section (30): C^*F_{cc}.--This matrix is obtained by multiplying F_{cc} by C^*. This computation is analogous to that explained in section (29), p. 61, with F_{bb} replaced by F_{cc}. The variance of c_i, the coefficient on z_i^*, (the ith element in the z* vector) is the element in the z_i^* row and z_i^* column of C^*F_{cc}. Like F_{cc}, C^*F_{cc} is a symmetrical matrix of order h. The nondiagonal elements of C^*F_{cc} are the covariance terms for the coefficients on the predetermined variables, the c's. For example, the element in the z_2^* row and z_3^* column of C^*F_{cc} is the covariance c_2c_3, where c_2 and c_3 are the coefficients on z_2^* and and z_3^*, respectively.

If covariance terms for the b's with the c's are desired, a matrix $C^*(-F_{bc})$ can be obtained by multiplying $-F_{bc}$ (the negative of the transpose of F'_{bc}, computed in section (25)) by C^*. This is a nonsymmetrical matrix with g-1 rows and h columns. The element in the y_i^* row and z_j^* column of $C^*(-F_{bc})$ is the covariance b_ic_j, where b_i and c_j are the coefficients on y_i^* and z_j^*, respectively.

The final step is to deadjust the coefficients and their standard errors and rewrite the equation in its original form, that is, in terms of one endogenous variable set equal to a linear combination of the other variables. This is described on p. 48.

Equations Having No z*'s.--For the special case in which z* has no elements, that is, h = 0, the method outlined in table 11 is followed with these changes: (a) Do not use sections (4), (5), (6) and (7); (b) section (8) is the same as section (1); and (c) do not use sections (24), (25), (26), (27), (28) and (30).

Procedures for Handling Complex Systems

In all systems of equations discussed in detail so far, each coefficient relates to a variable composed of a single item. Frequently, the coefficients in the structural equations relate to a composite variable that consists of an arithmetic sum of several endogenous or predetermined variables. Slight modifications are required when this is so. Other modifications may be needed when some of the equations can be fitted by least squares, but others are to be fitted by the limited information approach. Still further modifications are required when the system involves equations that are nonlinear in the variables. Methods of handling problems of this sort are discussed in detail in connection with the 3 examples included in this section.

A 6-equation Model of the Wheat Economy.--This example is taken from Meinken (16). This system of equations is designed to give simultaneously-determined estimates of prices in domestic and world outlets and the quantity of wheat utilized domestically in each of the following price-determined outlets: Use for food, feed, export, and end-of-year stocks. Total demand for use in these outlets plus that used for seed and industrial purposes must add to the total domestic supply at the beginning of the marketing year. The economic reasoning behind the equations used, and results from the statistical fitting, are discussed in detail in Meinken's bulletin. Aspects discussed here are those which require a different computational procedure from the systems of equations covered in previous sections.

The following variables are assumed to have been simultaneously determined by a common set of economic forces during the years included in the analysis. Thus, they are the endogenous variables in the structural equations.

P_w - Wholesale price of wheat at Liverpool, England, per bushel, converted to United States currency at par, cents; or P'_w - Wholesale price of wheat at Liverpool per bushel, converted to United States currency at current rate of exchange, cents.

P_d - Wholesale price of No. 2 Hard Red Winter wheat at Kansas City per bushel, cents.

C_f - Domestic use of wheat for feed, million bushels.

C_e - Domestic net exports and shipments to United States Territories of wheat and flour on a wheat equivalent basis, million bushels.

C_s - Domestic end-of-year stocks of wheat, million bushels.

C_h - Domestic use of wheat and wheat products for food by civilians, million bushels.

The following variables are believed to have influenced the values of 1 or more of the endogenous variables during the years included in the study but not to have been influenced by them to a significant degree during any given marketing year. Thus they are the predetermined variables in the structural equations.

S_w - World production of wheat plus stocks about August 1, excluding Russia and China but including net exports from Russia, million bushels.

I_w - Index of wholesale prices of 45 raw materials in England (1910-14 = 100).

S_d - Domestic production plus stocks on July 1 of wheat minus use for seed and industrial purposes, million bushels.

P_c - Wholesale price of No. 3 Yellow corn at Chicago, July-December, per 60-pound bushel, cents.

A - Poultry units fed on farms during the year beginning October, millions.

N - Ocean freight rates from Gulf ports to Liverpool plus the United Kingdom tariff on wheat from this country minus the average export subsidy paid by our government per bushel, cents.

I_d - Wholesale price of all commodities in this country as computed by the Bureau of Labor Statistics (1926 = 100).

F - Indications regarding the domestic winter wheat crop for the following year on December 1 as made by the Crop Reporting Board (planted acreage times condition).

L - Total population on January 1, millions.

Q - Average processing tax on wheat per bushel, cents.

M - Wage rates of all factory workers per hour, cents.

D - Consumers' disposable income, billion dollars.

T - Time (1921 = 1).

Symbolic letters should be used in connection with these "structural" variables, as the endogenous (Y_i) and predetermined (Z_i) variables used in fitting usually differ from those that appear in the structural equations.

Systems of equations may be fitted for at least 2 alternative reasons: (1) To determine certain structural coefficients, such as coefficients of elasticity, that are statistically consistent, or (2) to obtain simultaneous estimates of the several endogenous variables. If, in the example being considered in this section, the analyst were interested only in the elasticity of demand with respect to price for use of wheat as feed, only a single equation would need to be fitted. This would be fitted by the limited information approach, since the equation contains 2 endogenous variables. If the price of wheat were determined by a support program, the fitted equation could be used to estimate probable use of wheat for feed in future years. On the other hand, if simultaneous estimates of all of the endogenous variables in future years are desired, the number of equations used in fitting should equal the number of endogenous variables in the system.

Sometimes considerable algebraic manipulation is needed to reduce the equations to this number. This manipulation usually involves incorporating certain identities into the other equations. This should be done before determining the particular relations that are to be fitted by statistical means. If in the course of the algebraic manipulation certain structural coefficients are modified, these always can be derived algebraically from the coefficients of the fitted equations once the statistical analysis is complete. For the wheat model, the system as originally written included 6 equations as required for the 6 endogenous variables. The following equations are involved. The numbers are the same as in Meinken's bulletin 20/.

$$\frac{C_h}{L} = a_3 + b_{31}(P_d + Q) + b_{32}\frac{D}{L} + b_{33}T + b_{34}M \qquad (3.1)$$

$$C_f = a_4 + b_{41}(P_d - P_c) + b_{42}A \qquad (4)$$

$$C_e = a_5 + b_{51}(P_w' - P_d - N) \qquad (5)$$

$$C_s = a_6 + b_{61}\left(\frac{P_d}{I_d}\right) + b_{62}F \qquad (6)$$

$$S_d = C_h + C_f + C_e + C_s \qquad (7)$$

$$P_w = a_8 + b_{81}S_w + b_{82}I_w \qquad (8)$$

Equation (8) can be fitted directly by least squares as it contains only a single endogenous variable. But S_w and I_w should be included as predetermined variables in the system. P_w' rather than P_w is included in the set of equations to be fitted by the limited information approach 21/ (see equation (5)), but $P_w' = kP_w$. Hence, S_w and I_w each were multiplied by k to make them consistent with the variables used in the other equations.

Equation (7) need not be fitted as it involves no statistical coefficients. Hence, we have 4 equations, namely (3.1), (4), (5) and (6), to be fitted directly by the limited information method.

The predetermined variables for the entire system are obtained by picking out each predetermined variable included in any one of the structural equations. For this model, these are: L, Q, D/L, T, M, P_c, A, N, I_d, F, S_d, kS_w and kI_w.

20/ In equation (6) as given by Meinken (16), P_d/I_d is multiplied by 100. As this makes for a cumbersome notation, this constant is omitted in the equations as used in this handbook.

21/ If the disturbance in equation (8) is assumed to be independent of those in the other structural equations, then a least squares fit for this equation is equivalent to that obtained by the full information approach (see p. 28). If the disturbance is not assumed to be independent of those in the other equations, then a least squares fit is equivalent to that obtained by the limited information approach. Independence among the several disturbances in the structural equations normally is not assumed in the limited information approach.

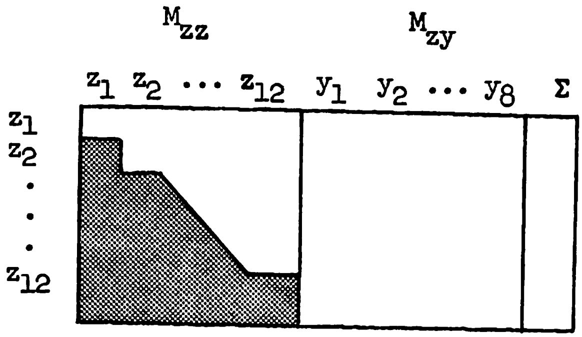
Since D does not appear separately in any equation, D/L can be used as a single variable. The 13 predetermined variables ordinarily would be designated as Z_1, Z_2, ... Z_{13}, respectively, and would be used to form the M_{zz} matrix of order 13. For this analysis, however, the variable T was not included in M_{zz} because this time trend is known to represent in only an imprecise way the true variables that cause the downward trend over time in the demand for wheat for use in food products. Thus M_{zz} became a matrix of order 12. To give determinate results, at least two more observations than the number of variables included in M_{zz} should be available 22/.

The endogenous variables to be used in fitting are obtained by using each combination of variables that includes at least one endogenous variable. In this connection, we are interested only in those endogenous variables that are included in equations to be fitted directly by the limited information approach. Hence, P_w is omitted. For this model, the endogenous variables used in fitting are: C_h/L, P_d+Q, C_f, P_d-P_c, C_e, P'_w-P_d-N, C_s, and P_d/I_d. These 8 variables are designated respectively as Y_1, Y_2, ... Y_8. Two of these Y_i appear in each of the 4 equations to be fitted by the limited information approach. Chernoff and Rubin (3, pp. 210-212) suggest that the nonlinear combinations C_h/L and P_d/I_d, as well as the linear combinations, be treated in this way. An alternative method of handling the nonlinear combinations of variables is discussed on p. 67.

In obtaining the moments, we use a symmetrical matrix of order 12 + 8 = 20, plus a check sum, making use of the variables other than T noted in the preceding two paragraphs. Some of the moments obtained are not used, but if only those moments actually to be used are obtained, with a check sum on each, more work is required than if the entire operation were to be performed with a single matrix. Moments that involve T with the other variables included in equation (3.1) (see p. 65) are obtained as a separate operation. (In some problems, as in the example described on p. 73, it is more efficient to obtain the moments with T simultaneously with those for the other variables, even though it is to be omitted from M_{zz}.)

The forward Doolittle solution required to obtain $M_{y*z}M_{zz}^{-1}M_{zy*}$ is next performed as a composite operation for all of the equations to be fitted by the limited information approach. The upper section of the forward Doolittle solution, in outline form, is as follows. Note that only a single check sum is used.

22/ Predetermined variables sometimes are omitted from M_{zz} to save computational time. Estimates of the coefficients that are statistically consistent still are obtained, provided sufficient predetermined variables are used to provide identification, but the estimates are less efficient (in a statistical sense) than if all of the predetermined variables in the system are used. (See Hildreth and Jarrett (12, pp. 69-70).)



The items from the last two rows of each section of the M_{zy} portion of this operation (see p. 32) are transferred to another sheet, with a check sum following each pair of y's. The boxheads on this sheet are of the form:

Row y_1 y_2 Σ y_3 y_4 Σ y_5 y_6 Σ y_7 y_8 Σ

The 4 $M_{y*z}M_{zz}^{-1}M_{zy*}$ required for the 4 equations are obtained as described on p. 32. Use of the several check sums provides a complete check on the operations. In some problems, it is more efficient to compute the complete $M_{yz}M_{zz}^{-1}M_{zy}$ matrix as a single operation, but in this example this method would require more than twice as many cumulative cross-multiplications.

In fitting each of the 4 equations, the following variables are involved:

Equation	Y*	Z*
(3.1)	C_h/L, $P_d + Q$	D/L, T, M
(4)	C_f, $P_d - P_c$	A
(5)	C_e, $P'_w - P_d - N$	-
(6)	C_s, P_d/I_d	F

These equations are fitted, making use of the appropriate Y's and Z's and the steps given in table 11 for equations having specified numbers of endogenous and predetermined variables. Each of these equations is overidentified. The M_{y*y*} matrices, which are of order 2 in each case, and the M_{z*y*} matrices are copied from the matrix of adjusted moments. In each equation, the variable to the left of the equality sign in the structural equation is designated as y_1 in the fitting process.

Nonlinear variables.--As noted on p. 66, Chernoff and Rubin (3, pp. 210-212) suggest treating nonlinear combinations that involve endogenous variables, such as C_h/L and P_d/I_d, as though they were each a single variable. Klein (13, pp. 120-121), on the other hand, suggests use of formulas that transform the nonlinear combinations into linear approximations. These approximations are then substituted for the original variables. The authors believe that the Klein approach is to be preferred.

The following formulas are given by Klein in this connection:

$$XY \simeq \bar{Y}X + \bar{X}Y - \bar{X}\bar{Y}$$

$$X/Y \simeq \bar{X}/\bar{Y} + X/\bar{Y} - (\bar{X}/\bar{Y}^2)Y.$$

where $\bar{X}$ and $\bar{Y}$ are the means of X and Y, respectively. If either the product or the quotient is multiplied by a constant, then each term in the transformation is multiplied by the constant.

Each of the nonlinear variables in the model for wheat involves a quotient. Hence these variables can be rewritten in the following way:

$$\frac{C_h}{L} \simeq \frac{\bar{C}_h}{\bar{L}} + \frac{1}{\bar{L}}C_h - \frac{\bar{C}_h}{\bar{L}^2} L$$

$$\frac{P_d}{I_d} \simeq \frac{\bar{P}_d}{\bar{I}_d} + \frac{1}{\bar{I}_d}P_d - \frac{\bar{P}_d}{\bar{I}_d^2} I_d$$

In computing M_{zz}, the predetermined variables $-(\bar{C}_h/\bar{L}^2)L$ and $-(\bar{P}_d/\bar{I}_d^2)I_d$ are substituted for L and I_d, respectively. The change from the M_{zz} used in the previous example can be made easily, as the variables L and I_d are, respectively, merely multiplied by a constant.

For purposes of fitting, the linear combination of endogenous and predetermined variables is used in the same way as the quotient in the preceding example. In using the structural equations for analytical purposes, however, it is convenient to rewrite these variables so that the endogenous and predetermined variables are separated (see p. 82). This can be done easily by an algebraic transformation.

Nonlinear equations.--All 6 equations used in the wheat model are stated in linear form. This is a desirable condition when using computational methods described in this handbook. 23/ In some models, the analyst may prefer to express his equations as linear in logarithms or as some other known function of the observed variables. This results in a system of equations that are consistent in so far as linearity is concerned so long as none of the equations are identities like equation (7) in the model for wheat. If identities of this sort are involved, all other equations must be linear in the actual variables if consistency among the several equations is to be maintained. If all the

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23/ Chernoff and Rubin (3, pp. 210-212) point out, in effect, that systems of equations that involve some equations for which the variables are expressed in logarithms and other equations for which the variables are in terms of actual data can be handled directly by the limited information approach by using the logarithms of the appropriate variables in M_{zz} in place of the original data. Estimates of the coefficients obtained in this way are statistically consistent but probably, under most conditions, are less efficient than if all variables are either in actual data or in logarithms.

equations are expressed as linear in logarithms, and nonlinear combinations of variables like those discussed in the preceding section are involved, no modification in the computational procedure is required as the logarithm of a product equals the sum of the logarithms of the factors and the logarithm of a quotient equals the logarithm of the numerator minus the logarithm of the denominator. Thus such combinations easily can be made linear as a part of the logarithmic transformation.

In connection with the wheat model, a logarithmic relation for equation (4), which deals with the demand for wheat for feeding, appears to prevail. However, this can be approximated by a linear relationship except in those years for which the spread between the price of wheat and corn is negative or exceeds about 40 cents per 60-pound bushel. This price spread exceeded 40 cents in 3 of the years used in fitting the equations. Use for feed in these years was considerably higher than would have been expected if based on a linear relation due to minimum requirements for wheat in poultry rations. A scatter diagram was made of this relationship and a free-hand line drawn. Use of wheat for feed in these years was read from the line and exports were increased by the difference between the quantity fed and the quantity indicated for feeding from the line. Thus the data used for fitting were transformed in such a way as to be consistent with a set of linear relations. A similar adjustment in the reverse direction is required when this relationship is used for analytical purposes. This adjustment is described in detail in the appendix of Meinken's bulletin.

A 9-equation Model of the Demand for Dairy Products.--Rojko (17) describes a number of 3-equation models of the demand structure for dairy products, including one for the post-World War II years. The following is a natural extension of his post-World War II model. As each regression coefficient relates to only a single variable, the variables are written down directly as Y's (endogenous) and Z's (predetermined). In the following listing, only enough information regarding the variables is given to indicate in a general way their economic meaning.

The following variables are involved in the model:

Y_1 - Quantity of milk used for fluid milk and cream
Y_2 - Price of fluid milk and cream
Y_3 - Quantity of milk used for butter
Y_4 - Price of butter
Y_5 - Quantity of milk used for American cheese
Y_6 - Price of American cheese
Y_7 - Quantity of milk used for manufactured products other than butter and American cheese
Y_8 - Price of manufactured products other than butter and American cheese
Y_9 - Quantity of margarine produced or sold
Z_1 - Disposable consumer income
Z_2 - Price of margarine, which is assumed to be determined by the fats and oils economy
Z_3 - Total quantity of milk available for consumption
Z_4 - Price of meat, poultry, and eggs

The following equations are used. In each case, the variable on the left is expressed as a linear function of those included in the parenthesis on the right. To save space, full detail for the equations is omitted.

$$
\begin{array}{lll}
Y_1 & = f(Y_2,\ Z_1) & (1) \\
Y_2 & = f(Z_1,\ Z_2,\ Z_3) & (2) \\
Y_3 & = f(Y_4,\ Z_1,\ Z_2) & (3) \\
Y_4 & = f(Z_1,\ Z_2,\ Z_3) & (4) \\
Y_5 & = f(Y_6,\ Z_1,\ Z_4) & (5) \\
Y_6 & = f(Z_1,\ Z_2,\ Z_3) & (6) \\
Y_7 & = f(Y_8,\ Z_1) & (7) \\
Y_8 & = f(Z_1,\ Z_2,\ Z_3) & (8) \\
Y_9 & = f(Y_4,\ Z_1,\ Z_2) & (9) \\
Z_4 & = Y_1 + Y_3 + Y_5 + Y_7 & (10)
\end{array}
$$

The economic reasoning behind these equations is discussed in detail in the article by Rojko (17).

This model has 10 equations and 9 endogenous variables. The 10 equations can be reduced to 9 by substituting the algebraic value of any one of the Y's in equation (10), for example Y_1, for its value in equation (1). This in no way affects the equation as these variables are algebraic equivalents. The chief importance of equation (10) is to remind the analyst that he cannot express his other equations as linear in the logarithms of the variables if all equations are to be consistent with respect to linearity.

In fitting these equations, Z_4 is omitted from M_{zz} for the same reason that time was omitted from M_{zz} for the model relating to wheat. That is, Z_4 is only an imprecise measure of prices of items that compete with American cheese as alternative sources of protein.

As each of the equations (2), (4), (6), and (8) involves only a single endogenous variable, they can be fitted directly by least squares. But as each involves all of the predetermined variables included in M_{zz}, much computational time can be saved by using the method described in succeeding paragraphs. Equation (10) need not be fitted. Each of the remaining equations is overidentified; after the computations described below are completed, they are handled by the methods outlined in table 11.

In obtaining the moments, a symmetrical matrix of order 3 + 9 = 12 is used, plus a check sum. (Moments involving Z_4 are computed separately.) The forward Doolittle solution is used to obtain (1) M_{zz}^{-1} (by making use of an identity matrix) and (2) $M_{y^*z}M_{zz}^{-1}M_{zy^*}$ as a composite operation. The upper section of the forward Doolittle solution, in outline form, is as follows. Note that only a single over all check sum is used.

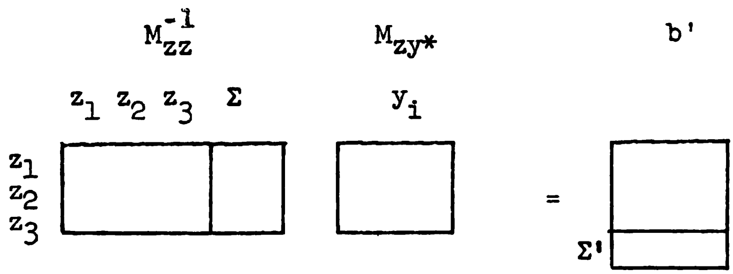
	M_{zz}			M_{zy}				I			
	z_1	z_2	z_3	y_1	y_2	...	y_9	I_1	I_2	I_3	Σ
z_1								1	0	0	
z_2								0	1	0	
z_3								0	0	1	

The items from the last two lines in each section of the M_{zy} and the I portions of this operation (see p. 32) are transferred to another sheet, with check sums computed in the following way:

$$y_1\ y_2\ \Sigma\ y_3\ y_4\ \Sigma\ y_5\ y_6\ \Sigma\ y_7\ y_8\ \Sigma\ y_9\ y_4\ \Sigma\ I_1\ I_2\ I_3\ \Sigma$$

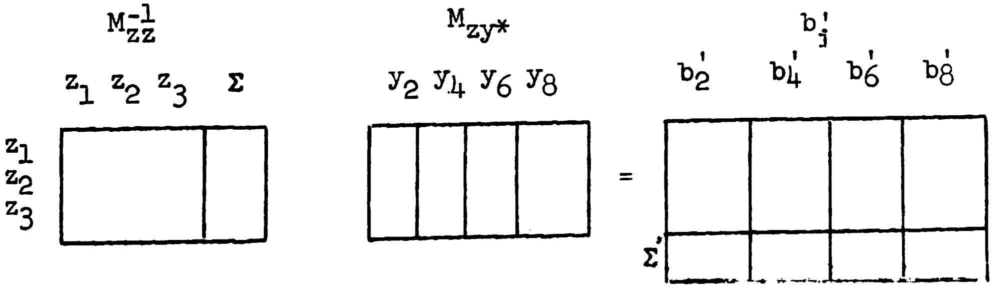
The 5 $M_{y*z}M_{zz}^{-1}M_{zy*}$ required for the 5 equations to be fitted by the limited information approach are obtained as described on p. 32. Use of the several check sums provides a complete check on the operation. M_{zz}^{-1} likewise is obtained in the way described on p. 26. **The remaining operations for the overidentified equations are the same as for any system.**

For each of the equations to be fitted by least squares, we first obtain $b' = M_{zz}^{-1}M_{zy*}$. For each equation, M_{zy*} is a vector, as only 1 y* is involved. Since M_{zz}^{-1} is symmetrical, this multiplication can be carried out either as a row-by-column or as a column-by-column product; the column-by-column product (see p.101) may be preferred as a matter of convenience and the checks shown in the outline given below are designed for a multiplication performed in this way. The missing elements in M_{zz}^{-1} should be filled in before performing the multiplication. The sum of the 3 b_{ij} in the b' vector should approximately equal the sum of the products obtained by making use of the check sum column. This checks the computation. This computation in outline form is as follows:



The elements of the b' vector are the respective highest-order partial regression coefficients for the equation that involves y_i.

These computations for the 4 equations to be fitted by least squares can be systematized in the following way:



In this computation, the product $M_{zz}^{-1}M_{zy*}$ can be obtained conveniently for each equation in turn. Here the Mzy* and b'_i sections each consist of 4 separate column vectors. Each set of computations is the same as that described in the preceding paragraph.

The remaining statistical coefficients for these equations are next obtained, again making use of a composite set of computations. To fill in the first two columns in the following outline, use is made of values obtained in fitting other equations by the limited information approach. For example, $M_{y_i y_i}$ for y_2 is the element in the lower right corner of $M_{y^* y^*}$ for equation (1), and $M_{y_i z} M_{zz}^{-1} M_{z y_i}$ for y_2 is the element in the lower right corner of $M_{y^* z} M_{zz}^{-1} M_{z y^*}$ for equation (1). A similar situation holds for y_4 and the matrices for equation (3), y_6 and the matrices for equation (5), and y_8 and the matrices for equation (7). N, the number of observations in the sample, and N', the number of observations minus the number of variables, are the same for each equation. For this example, N' = N - 4. The outline below gives the value of the multiple coefficient of determination and the standard error of estimate for each equation. A check on the copying of data in the first two columns is given by computing the items in column (4) and checking them against the appropriate elements of $W_{y^* y^*}$ for each equation. These should check exactly. The other items are checked by recomputation.

y_i	(1) $M_{y_i y_i}$	(2) $M_{y_i z} M_{zz}^{-1} M_{z y_i}$	(3) $R_i^2 = \frac{(2)}{(1)}$	(4) $W_{y_i y_i}$ = (1) - (2)	(5) $s_i^2 = \frac{(4)}{NN'}$	(6) $s_i = \sqrt{(5)}$
y_2						
y_4						
y_6						
y_8						

The standard errors of the regression coefficients for each equation are obtained by making use of the following outline. The items in column (1) are obtained by multiplying by N the corresponding items in column (5) of the preceding outline. If desired, the diagonal elements from M_{zz}^{-1} can be written above the box heads so as to be conveniently available in performing these operations. The diagonal elements from M_{zz}^{-1} are multiplied, respectively, by the items in column (1), and the square root of the product is obtained. The items in column (3) are the standard errors of the highest order partial regression coefficients between y_i and z_1; the items in column (5) are the standard errors of the regression coefficients between y_i and z_2, and so forth. The check is one of recomputation.

		$z_{11}^{-1}=$		$z_{22}^{-1}=$		$z_{33}^{-1}=$	
y_i	$W_{y_i y_i}/N'$ = Ns_i^2	$z_{11}^{-1} \times (1)$		$z_{22}^{-1} \times (1)$		$z_{33}^{-1} \times (1)$	
		Value	Square root	Value	Square root	Value	Square root
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
y_2							
y_4							
y_6							
y_8							

The regression coefficients and their standard errors and the standard error of estimate must be deadjusted and constant terms obtained for the equations. The procedure for each equation is exactly the same as shown in the lower section of table 3. When this approach is used, coefficients of partial determination cannot be obtained conveniently.

The method of obtaining these coefficients from computations used in connection with fitting equations by the limited information approach is derived from material given in Hildreth and Jarrett (12, pp. 147-151). It should be remembered that this method can be used directly only when M_{zz} is the same for both sets of equations.

A Partially-Reduced Form Model for Feeds.--The system of equations described in this section was developed by Gordon King, Agricultural Marketing Service, United States Department of Agriculture, to study effects of the price support program for feed grains on prices and use for feeding of feed concentrates. As in the preceding example, only enough information regarding the variables is given to indicate in a general way their economic meaning.

The following endogenous variables are involved in the model:

P_g - Price of feed grains that actually prevailed

P_h - Price of high protein feeds

P_f - Price of feed grains that would have prevailed had there been no price support program for feed grains

Q_1 - Quantity of feed grains going under loan and remaining under loan at the end of the marketing period plus grain delivered under purchase agreements

The following predetermined variables are involved in the model:

Q_g - Supply of feed grains available for feeding during the marketing period

Q_h - Quantity of high protein feeds used for feed

P_s - Support price for corn (In years when no support program was in operation, this variable was set at such a level that no grain would normally have gone under loan.)

A - Grain-consuming animal units fed during the year

P_1 - Price of livestock and livestock products

T - Time

In a more complete model, A and P_1, and possibly Q_h, would have been treated as endogenous variables.

The following structural equations are involved. In each case, the variable on the left is expressed as a linear function of those included in the parenthesis on the right. To save space, full detail for the equations is omitted.

$$P_f = f(Q_g, Q_h, A, P_1) \qquad (1)$$

$$Q_1 = f(P_f - P_s) \qquad (2)$$

$$P_g = f(Q_g - Q_1, Q_h, A, P_1) \qquad (3)$$

$$P_h = f(Q_g - Q_1, Q_h, A, P_1, T) \qquad (4)$$

As no data are available for P_f for years in which a support program was in operation, the right-hand side of equation (1) is substituted for P_f in equation (2). The resulting equation, shown here as equation (5), is called a partially-reduced form equation.

$$Q_1 = f(Q_g, Q_h, A, P_1, -P_s) \qquad (5)$$

Equations (3), (4), and (5) are used in fitting. The following variables are used: $Y_1 = P_g$, $Y_2 = Q_g - Q_1$, $Y_3 = P_h$, $Y_4 = Q_1$, $Z_1 = Q_g$, $Z_2 = Q_h$, $Z_3 = -P_s$, $Z_4 = A$, $Z_5 = P_1$, $Z_6 = T$. The following tabulation shows the Y* and Z* involved in each equation:

Equation	Y*	Z*
(3)	Y_1, Y_2	Z_2, Z_4, Z_5
(4)	Y_3, Y_2	Z_2, Z_4, Z_5, Z_6
(5)	Y_4	Z_1, Z_2, Z_3, Z_4, Z_5

As equation (5) involves only a single endogenous variable, it can be fitted directly by least squares. As in the preceding example, however, computational time is saved by handling all computations as a composite unit. The counting rule does not apply strictly to equations (3) and (4) because each contains a composite variable, but if the rule were applied to the equations when they are expressed in Y's and Z's it would suggest that they are over-identified, which is correct. In the model described in succeeding paragraphs, Z_6 (which represents a time trend) is omitted from M_{zz}. In other formulations, however, Z_6 might be included in M_{zz}, as the underlying time trend is believed to be nearly linear over the period included in the analysis.

In setting up the forward Doolittle solution, we consider three sorts of computations: Those that relate to obtaining (a) $M_{y^*z}M_{zz}^{-1}M_{zy^*}$ for equations (3) and (4); (b) M_{zz}^{-1} for equation (5); and (c) $M_{z^*z^*}^{-1}$ for equations (3) and (4). As in the preceding example, the independent variables in equation (5) are the same as the predetermined variables in M_{zz} for the system. Also 3 of the predetermined variables in equations (3) and (4) are identical. These 3 variables are listed first in the forward Doolittle solution and then the others that are involved in M_{zz}. All 4 y's are next listed, and then the 5-variable identity matrix. As in the preceding example, a single check sum is used. The upper section of the forward Doolittle solution has the following form:

	M_{zz}					M_{zy}				I					
	z_2	z_4	z_5	z_1	z_3	y_1	y_2	y_3	y_4	I_2	I_4	I_5	I_1	I_3	Σ
z_2										1	0	0	0	0	
z_4										0	1	0	0	0	
z_5										0	0	1	0	0	
z_1										0	0	0	1	0	
z_3										0	0	0	0	1	

By computing a new check sum for the last two rows of each section of the forward Doolittle solution, M_{zz}^{-1} can be obtained and checked directly from the I part of the solution. The inverse should be written down on a second sheet, with space at the right for additional columns. Missing elements from this symmetrical matrix should be filled in. The vector Mzy_4 can be written in an adjacent column and the highest order b's obtained directly by performing a column-by-column multiplication as described on p. 71. As only a single equation is to be computed by least squares, R^2 and s can be computed directly by use of the formulas implied by the table on p. 72. $My_4zM_{zz}^{-1}Mzy_4$ can be obtained by cumulating the products from the last two rows of each section of the y_4 column of the forward Doolittle solution in the way described on p. 32. Squared values of the standard errors of the respective regression coefficients are obtained by multiplying the diagonal elements of M_{zz}^{-1} by Ns^2. These can be written down directly in a column next to the b's, and the square roots inserted in the next column. Except for deadjusting the coefficients, this completes the computations for the least squares equation.

Because of the nature of the y's involved in equations (3) and (4), it is efficient to compute the entire $M_{yz}M_{zz}^{-1}M_{zy}$ matrix for y_1, y_2, and y_3. This can be done by placing a strip of paper over the y_4 column and computing a new check sum for y_1, y_2 and y_3 for the last two rows of each section of the forward Doolittle solution. The triple-product matrix then is obtained and checked as described on p. 32. At the same time, computations involved in obtaining M_{z*z*}^{-1} for these equations can be completed. This can be done by placing strips of paper over the forward Doolittle solution in such a way that columns for z_1, z_3, y_1, y_2, y_3, y_4, I_3, and the original Σ, and rows and sections for z_1 and z_3 are covered. Data for z_6 should be inserted in place of those originally shown for z_1, and separate check sums are computed for the z's and the I's that appear on the modified table. By completing the indicated operations for z_6 and the new check sums, information needed to obtain the M_{z*z*}^{-1} is given. The elements of the inverse for equation (3) can be obtained as a part of the computations involved in obtaining the elements of the inverse for equation (4). The check on the operation for equation (4) automatically checks the computation for equation (3).

These operations having been completed, the usual computations for an overidentified equation are carried out for each equation as shown in table 11. If desired, M^{-1}_{z*z*} can be recorded directly in the space provided for step four in each table. In carrying out the computations for equation (4), it should be remembered that the y variables are listed in the order y_3, y_2.

TESTS TO BE MADE AFTER COMPLETING THE ANALYSIS

Four tests are commonly made by an analyst after he has run any correlation or regression study. (1) The signs and relative magnitudes of the regression coefficients are checked to make sure that they are consistent with the analyst's expectations based on economic theory and knowledge of the field to which the study relates. (2) The coefficients are compared with their standard errors to see whether they differ from zero or some other value by a statistically significant amount. (3) The unexplained residuals are analyzed, perhaps by plotting them in a time series, to see whether they are essentially random. (4) If the study is an important one, the analyst makes up charts that indicate the degree of partial correlation and studies these to learn how particular observations affect the analysis and to see if the type of function fitted appears to be consistent with the data. Tests of this sort as applied to least squares analyses are described briefly by Foote and Fox (9, pp. 25-36). The analysis also would be checked to see how useful it is for prediction for observations not included in the study.

The first of these checks is equally important when working with systems of equations. Before using the system for analytical purposes, the analyst must make up his mind whether any gross inconsistencies render the results entirely useless for such purposes. In some situations, he may be content to replace certain coefficients with others that on a judgment basis appear more logical. In so doing, however, calculated standard errors for other coefficients lose their validity. As noted on p. 1, t-tests of statistical coefficients to indicate whether they differ significantly from some assumed value, normally zero, may not be valid when the analyst is working with the kind of time series normally used in economic research; but the ratio of the estimate to its calculated standard error ordinarily should give a rough idea of the probable sampling variability of the estimate. A test that indicates the probability that the unexplained residuals are serially correlated is described in the following section. This test was designed for use with equations fitted by least squares but on intuitive grounds it appears to be fairly applicable for use with equations fitted by the limited information approach. Following that, tests are described that indicated whether particular equations fitted by the limited information approach are, in fact, probably overidentified or just identified. Graphic studies of the individual equations usually are not made when we are working with systems of equations. Detailed studies of the usefulness of the system for projecting trends or analyzing alternative policy questions represent a partial substitute. Formal tests of significance of the residuals to check whether the estimated relations fit data outside the sample period as well as should be expected from their fit during the sample period are described by Hildreth and Jarrett (12, pp. 119-129). These tests are not described here; they are only a "crude makeshift" as applied to equations fitted by the limited information approach.

Serial Correlation in the Residuals

Durbin and Watson (6) developed a method by which the unexplained residuals from an equation fitted by least squares can be tested to see if successive values are correlated. 24/ This sort of correlation is commonly referred to as "serial" correlation. Use of the limits shown in table 14 must be regarded as approximate when this test is applied to residuals from equations fitted by the limited information approach or to equations fitted by least squares that contain a lagged endogenous variable. But no exact test is available for such residuals.

In using this test, we compute the following statistic:

$$d' = \frac{\sum_{t=2}^{N} (d_t - d_{t-1})^2}{\sum_{t=1}^{N} d_t^2}$$

where d_t is the unexplained residual for observation t. If gaps occur in the data, the number of observations that enter into the numerator are reduced by one for each gap. This statistic can be computed for equations fitted by least squares or for each structural equation in a system of simultaneous equations.

Table 14 is used to obtain upper and lower bounds for the critical values for a 2-tailed test at a 5-percent probability level. N is the number of observations in the analysis and k' is the number of independent or predetermined variables in the equation. 25/ We compute d' and 4 - d' and find the appropriate value for d_L and d_U in the table. The two computed values relate to the two tails of the sampling distribution, d' relating to positive serial correlation and 4-d', to negative serial correlation. If d' or 4-d' is less than d_L, we assume that the residuals may be serially correlated, either positively or negatively. If both d' and 4-d' are greater than d_U, we assume that there is no serial correlation. If neither of the computed values is less than d_L, but one of them lies between d_L and d_U, the test is inconclusive.

24/ Klein (13, pp. 89-90) suggests, in this connection, use of a test developed by Hart and von Neumann (11) in 1942. This test, however, relates to serial correlation in a variable rather than to serial correlation in the unexplained residuals as such. The Durbin-Watson test was not published until 1951. On intuitive grounds, it appears to be preferred as a test for residuals.

25/ This procedure is suggested by Hildreth and Jarrett (12, p. 78) based on an analogy between limited information and least squares estimates. Professor Durbin, in a discussion of a paper by Morgan and Corlett in the Journal of the Royal Statistical Society, 64:355, suggests instead use for N of the number of observations minus the number of dependent variables less one. He says, "For the overidentified case it seems possible to obtain inequalities for the significance points".

Table 14.- Significance points of d_L and d_U

N	k'=1		k'=2		k'=3		k'=4		k'=5	
	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U
15	0.95	1.23	0.83	1.40	0.71	1.61	0.59	1.84	0.48	2.09
16	0.98	1.24	0.86	1.40	0.75	1.59	0.64	1.80	0.53	2.03
17	1.01	1.25	0.90	1.40	0.79	1.58	0.68	1.77	0.57	1.98
18	1.03	1.26	0.93	1.40	0.82	1.56	0.72	1.74	0.62	1.93
19	1.06	1.28	0.96	1.41	0.86	1.55	0.76	1.72	0.66	1.90
20	1.08	1.28	0.99	1.41	0.89	1.55	0.79	1.70	0.70	1.87
21	1.10	1.30	1.01	1.41	0.92	1.54	0.83	1.69	0.73	1.84
22	1.12	1.31	1.04	1.42	0.95	1.54	0.86	1.68	0.77	1.82
23	1.14	1.32	1.06	1.42	0.97	1.54	0.89	1.67	0.80	1.80
24	1.16	1.33	1.08	1.43	1.00	1.54	0.91	1.66	0.83	1.79
25	1.18	1.34	1.10	1.43	1.02	1.54	0.94	1.65	0.86	1.77
26	1.19	1.35	1.12	1.44	1.04	1.54	0.96	1.65	0.88	1.76
27	1.21	1.36	1.13	1.44	1.06	1.54	0.99	1.64	0.91	1.75
28	1.22	1.37	1.15	1.45	1.08	1.54	1.01	1.64	0.93	1.74
29	1.24	1.38	1.17	1.45	1.10	1.54	1.03	1.63	0.96	1.73
30	1.25	1.38	1.18	1.46	1.12	1.54	1.05	1.63	0.98	1.73
31	1.26	1.39	1.20	1.47	1.13	1.55	1.07	1.63	1.00	1.72
32	1.27	1.40	1.21	1.47	1.15	1.55	1.08	1.63	1.02	1.71
33	1.28	1.41	1.22	1.48	1.16	1.55	1.10	1.63	1.04	1.71
34	1.29	1.41	1.24	1.48	1.17	1.55	1.12	1.63	1.06	1.70
35	1.30	1.42	1.25	1.48	1.19	1.55	1.13	1.63	1.07	1.70
36	1.31	1.43	1.26	1.49	1.20	1.56	1.15	1.63	1.09	1.70
37	1.32	1.43	1.27	1.49	1.21	1.56	1.16	1.62	1.10	1.70
38	1.33	1.44	1.28	1.50	1.23	1.56	1.17	1.62	1.12	1.70
39	1.34	1.44	1.29	1.50	1.24	1.56	1.19	1.63	1.13	1.69
40	1.35	1.45	1.30	1.51	1.25	1.57	1.20	1.63	1.15	1.69
45	1.39	1.48	1.34	1.53	1.30	1.58	1.25	1.63	1.21	1.69
50	1.42	1.50	1.38	1.54	1.34	1.59	1.30	1.64	1.26	1.69
55	1.45	1.52	1.41	1.56	1.37	1.60	1.33	1.64	1.30	1.69
60	1.47	1.54	1.44	1.57	1.40	1.61	1.37	1.65	1.33	1.69
65	1.49	1.55	1.46	1.59	1.43	1.62	1.40	1.66	1.36	1.69
70	1.51	1.57	1.48	1.60	1.45	1.63	1.42	1.66	1.39	1.70
75	1.53	1.58	1.50	1.61	1.47	1.64	1.45	1.67	1.42	1.70
80	1.54	1.59	1.52	1.62	1.49	1.65	1.47	1.67	1.44	1.70
85	1.56	1.60	1.53	1.63	1.51	1.65	1.49	1.68	1.46	1.71
90	1.57	1.61	1.55	1.64	1.53	1.66	1.50	1.69	1.48	1.71
95	1.58	1.62	1.56	1.65	1.54	1.67	1.52	1.69	1.50	1.71
100	1.59	1.63	1.57	1.65	1.55	1.67	1.53	1.70	1.51	1.72

From Durbin and Watson (6, p. 174). Reproduced by permission of Biometrika.

Overidentifying Restrictions

Anderson and Rubin (1, p. 56) developed a test designed to determine the .alidity of assumptions that certain variables do not appear in a particular equation, given the validity of the remaining specifications regarding the system. The number of variables in a system that are omitted from a particular equation determine the degree of identification; hence this is known as a test of overidentifying restrictions. As noted by Hildreth and Jarrett (12, p. 79), in many applications to economics the investigator is likely to have better grounds for the specification of which variables enter particular equations than for other aspects of the system, such as, for example, whether the relations are linear in actual data or in logarithms. In such cases the interpretation of the test is subject to doubt, but it remains true that an extreme value for the test statistic is an indication of difficulty somewhere in the statistical specification used. Because of the nature of the test, it is used only for overidentified equations.

The test is carried out in 2 stages. First we determine whether it is reasonable to conclude that the equation is overidentified. We do this by postulating a null hypothesis that the equation is underidentified or just identified. If this null hypothesis is not rejected, we make a second test to determine whether it is reasonable to conclude that the equation is just identified. The first test can be made easily from data computed in the fitting process; the second test requires a considerable number of additional computations whenever the number of y* exceeds 2.

First Phase.--To carry out the first phase of the test, we compute the following:

$$(2.3026)N \log_{10} (1 + 1/\lambda)$$

where N is the sample size, and λ was computed in section (9) (see p. 53). This statistic follows approximately a χ^2 distribution with $H - h - g + 1$ degrees of freedom, where H is the total number of predetermined variables in the system, h is the number of predetermined variables in the equation, and g is the number of endogenous variables in the equation. If the computed test statistic exceeds the critical value associated with the number of degrees of freedom at the desired probability level, we have reason to conclude that the equation is overidentified. Tables of expected values of χ^2 when the assumed hypothesis is true are given in most textbooks on statistics and elsewhere. Probabilities of 5 percent or 1 percent are those commonly accepted by statisticians.

Second Phase.--If the computed test statistic for the first phase is less than the critical value, we proceed with the second phase. The value of λ used in the first stage of the test is the largest root of a certain equation. For the second stage of the test, we need to determine the second largest root of the same equation. This can be done by transforming the original matrix in the way described below and then getting the largest root of the transformed matrix by the iterative process described on p. 55. However, if $g = 2$, the second largest root can be determined by a direct solution of the equation. The paragraphs that follow indicate the way in which the second largest root is determined for a g of specified size. Absolute values of the roots are used in each phase.

(1) $g = 2$: The second largest root is given directly by the formula

$$\lambda_2 = \frac{p_2 - \sqrt{p_2^2 - 4p_1p_3}}{2p_3}$$

where the p's are defined as in step (9) (see p. 38). For equation (2.1) of the lumber problem, application of this formula gives a value for the second largest root of 0.4550.

(2) $g \geq 3$: Use is made of results obtained by the iterative process described on p. 55. The example used for illustrative purposes in table 13 is that of equation (2.1) of the lumber problem, for which $g = 2$. This example also will be used in this section, although the reader should note that in actual practice this approach is used only when $g \geq 3$. Modifications required for a larger number of y^* are obvious.

The first step is to examine the last column for q shown in table 13 to determine the row in which the 1 occurs. For this example, this is found in column $q^{(6)}$ and the 1 occurs in the first row. This tells us that in the next step we make use of the first column of the A' matrix originally used in table 13. Had the 1 appeared in the second row, we would have used the second column from the A' matrix. This column vector is multiplied by the transpose of the column vector for q in the last column of table 13. For this example, the computation is as follows:

$$\begin{bmatrix} 1.7305 \\ -8.6573 \end{bmatrix} \begin{bmatrix} 1 & -0.8555 \end{bmatrix} = \begin{bmatrix} 1.7305 & -1.4805 \\ -8.6573 & 7.4066 \end{bmatrix}$$

The resulting product matrix then is subtracted from A' to give V. The computation is as follows:

$$\begin{bmatrix} 1.7305 & -1.0912 \\ -8.6573 & 7.8617 \end{bmatrix} - \begin{bmatrix} 1.7305 & -1.4805 \\ -8.6573 & 7.4066 \end{bmatrix} = \begin{bmatrix} 0 & 0.3893 \\ 0 & .4550 \end{bmatrix}$$

A partial check on the computation is given by the fact that a column of zeros always must be found in V in the column corresponding to the row in which the 1 occurred in the last column of table 13.

We now obtain V' by interchanging rows and columns of V and operate on V' in table 13 in exactly the same way as we did on the A' matrix. When $g = 2$, only a single iteration is required. The second column for Q is as follows:

$$\begin{bmatrix} 0.3893 \\ .4550 \end{bmatrix}$$

This indicates that the second largest root equals 0.4550, which is the same as obtained by use of the formula.

To carry out the second phase of the test, we compute the following:

$$(2.3026)N \log_{10}(\lambda_1 + \lambda_2 - 2)$$

where N is the sample size, λ_1 is the largest root found in section (9) and λ_2 is the second largest root found by the computation described above. This statistic follows approximately a χ^2 distribution with H - h - g + 2 degrees of freedom. If the computed test statistic exceeds the critical value associated with the number of degrees of freedom at the desired probability level we have reason to conclude that the equation is just identified. If the computed test statistic is less than this value, the test suggests that the equation is underidentified and little confidence can be placed in the values obtained for the coefficients.

USE OF SIMULTANEOUS EQUATION MODELS FOR ANALYTICAL PURPOSES

After the coefficients in a complete system of equations have been obtained by appropriate statistical means, the analyst will in general wish to use the system for predicting trends or for other analytical purposes. For this use, the approach is different from that employed when a single equation is used, although the same general steps are involved. In this section, we first discuss an approach that can be used for any system of equations, and then describe some computational shortcuts that frequently can be used. In the general approach, the n equations in n endogenous variables must be solved for each period for which an estimate is desired, making use of the coefficients estimated by the fitting process. In this connection, the reader should remember that one of the reasons for using a system of simultaneous equations is to ensure that estimates of the endogenous variables made from the system are mutually consistent. The 6-equation model for wheat serves as a convenient example for the general solution even though this approach would not be used in connection with actual computations (see p. 84). We first consider a general solution for any system of simultaneous equations.

Solution of Simultaneous Equations

Consider a system of equations of the following form:

$$\begin{aligned}
b_{11}X_1 + b_{12}X_2 + \dots b_{1n}X_n &= a_1 \\
b_{12}X_1 + b_{22}X_2 + \dots b_{2n}X_n &= a_2 \\
&\vdots \\
b_{n1}X_1 + b_{n2}X_2 + \dots b_{nn}X_n &= a_n
\end{aligned}$$

Since we have n equations in n unknowns, in general a unique solution to this system of equations will exist.

If B is an n by n matrix of the b_{ij}, X' is a column vector of the X_j, and A' is a column vector of the a_i, this system of equations can be written in the following matrix notation, BX' = A'. The truth of this is immediately evident if we consider any element in the product vector.

We now multiply each side of the matrix equation by B^{-1}. This gives

$$B^{-1}BX' = B^{-1}A'$$

or

$$B^{-1}A' = X'$$

Translating the matrix notation back into terms of ordinary algebra, we have the following system of equations that can be used to solve for the X's:

$$\begin{aligned}
b_{11}^{-1}a_1 &+ b_{12}^{-1}a_2 + \dots b_{1n}^{-1}a_n = X_1 \\
b_{21}^{-1}a_1 &+ b_{22}^{-1}a_2 + \dots b_{2n}^{-1}a_n = X_2 \\
&\vdots \\
b_{n1}^{-1}a_1 &+ b_{n2}^{-1}a_2 + \dots b_{nn}^{-1}a_n = X_n
\end{aligned}$$

For most systems of equations, the b's are given and we wish to determine the X's associated with a particular set of a's. This can be done easily by making use of the second system of equations based on the elements of the inverse matrix.

Use of Reduced Form Equations

Let us rewrite the structural equations for the model for wheat (see p. 65) as shown below. In doing so, we assume that a value for P_w is obtained directly from equation (8) in the same way as for any least squares analysis, and that this is multiplied by the appropriate value of k to give P_w'. This variable is then treated in the equations used for analytical purposes as though it were predetermined.

$$C_h/L - b_{31}P_d = a_3 + b_{31}Q + b_{32}(D/L) + b_{33}T + b_{34}M = A_3 \qquad (3.1)$$

$$C_f - b_{41}P_d = a_4 - b_{41}P_c + b_{42}A = A_4 \qquad (4)$$

$$C_e + b_{51}P_d = a_5 + b_{51}(P_w' - N) = A_5 \qquad (5)$$

$$C_s - b_{61}(P_d/I_d) = a_6 + b_{62}F = A_6 \qquad (6)$$

$$C_h + C_f + C_e + C_s = S_d = A_7 \qquad (7)$$

The A's equal the sum of the predetermined variables in each equation multiplied by their appropriate coefficients plus the constant term. In general, the A's differ in each period.

Since we have 5 equations in 5 unknowns, ordinarily we can solve for the values of the 5 endogenous variables in any given year by making use of the inverse of the matrix of coefficients of the endogenous variables and an appropriate vector of the A's. The matrix of coefficients of the endogenous variables equals the following:

$$\begin{bmatrix} 1/L & 0 & 0 & 0 & -b_{31} \\ 0 & 1 & 0 & 0 & -b_{41} \\ 0 & 0 & 1 & 0 & b_{51} \\ 0 & 0 & 0 & 1 & -b_{61}/I_d \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

This is a nonsymmetrical matrix. Methods of inversion for matrices of order greater than 2 described so far in this handbook apply only to symmetrical matrices. As shown in the next section, we do not need to obtain the inverse of this matrix for this example because we can derive the needed formulas directly from the equations themselves. Methods that could be used to invert this matrix are described in the appendix, p. 98.

This matrix illustrates one objection to the use of nonlinear endogenous variables in connection with systems of equations. All of the elements of the matrix are constants except for the coefficients that relate to the 2 nonlinear variables. If the inverse of the matrix of coefficients had to be used for this system, a new inverse for each year would be required. Had we used all linear variables, or the linear approximation for the nonlinear variables described on p. 68, the matrix of coefficients for the (linearized) endogenous variables would have consisted only of constants, and only a single inversion would be required. Nonlinear predetermined variables present no problem, as the A's in any case are recomputed for each year.

Given the inverse of this matrix, each endogenous variable can be expressed as a linear function of the A's multiplied by appropriate elements of the inverse. If D represents the matrix of coefficients, these equations have the following form:

$$C_h = d_{11}^{-1}A_3 + d_{12}^{-1}A_4 + d_{13}^{-1}A_5 + d_{14}^{-1}A_6 + d_{15}^{-1}A_7$$

$$C_f = d_{21}^{-1}A_3 + d_{22}^{-1}A_4 + d_{23}^{-1}A_5 + d_{24}^{-1}A_6 + d_{25}^{-1}A_7$$

etc.

In these, each endogenous variable is in effect expressed as a function of all of the predetermined variables in the system. These are known as the reduced form equations. They represent a convenient way to express the equations to readers not acquainted with matrix notation, particularly when the elements of the inverse are replaced by a set of actual numbers.

A new set of reduced form equations must be algebraically determined whenever changes are made in the basic structure of the model, as, for example, when an endogenous price is replaced by a predetermined one based on a price support program.

Simplified Formulas for the Wheat and Dairy Models

In the case of both the wheat and dairy models, simple formulas for simultaneously projecting the endogenous variables can be developed directly from the structural equations as such. In some cases, similar methods can be used to reduce the size of the matrix of coefficients to be inverted. To illustrate this point, a modified version of the wheat model is presented in the last part of this section. Frequently this general approach can be used in working with the kinds of equations that are common in economic analysis.

We first consider the equations for wheat presented on p. 82, after substituting the A's for the terms that involve only predetermined variables. If each term in equation (3.1) is multiplied by L, the following system of equations is given:

$$\begin{aligned}
C_h \qquad\qquad\qquad\qquad & - b_{31}LP_d &= A_3L \\
C_f \qquad\qquad\qquad & - b_{41}P_d &= A_4 \\
C_e \qquad\qquad & + b_{51}P_d &= A_5 \\
C_s \qquad & - b_{61}(P_d/I_d) &= A_6 \\
C_h + C_f + C_e + C_s & &= A_7
\end{aligned}$$

If each of the first 4 equations is subtracted from the last equation, all of the endogenous variables except P_d are eliminated. P_d then can be expressed as the following function of L, I_d, the A's, and the b's:

$$P_d = \frac{A_7 - A_3L - A_4 - A_5 - A_6}{Lb_{31} + b_{41} - b_{51} + (b_{61}/I_d)}$$

Transposition of the first 4 equations yields the following:

$$\begin{aligned}
C_h &= L(b_{31}P_d + A_3) \\
C_f &= b_{41}P_d + A_4 \\
C_e &= -b_{51}P_d + A_5 \\
C_s &= (b_{61}/I_d)P_d + A_6
\end{aligned}$$

The way in which these equations are used as applied to actual projections is discussed in detail by Meinken (16).

Use of the structural equations in the dairy model for analytical purposes is even simpler. Here we estimate Y_2, Y_4, Y_6, and Y_8 directly from the given values of the predetermined variables for the particular year, making use of equations (2), (4), (6), and (8). When these values and those for the predetermined variables are substituted in the remaining structural equations, values for the remaining Y's are given directly. Equation (10) provides a partial check on the computations.

In connection with analyses of the effect of alternative support programs on prices for wheat and corn, it is useful to modify the structural equations for wheat to include the price of corn as an endogenous variable. The way in which the additional equation is derived need not concern us here. This example, however, provides a useful addition to those discussed in preceding paragraphs in indicating useful techniques for deriving formulas for projection. In this example we are concerned with the following structural equations, where the A's have the same meaning as in the preceding examples:

$$\begin{aligned}
C_h \qquad\qquad\qquad\qquad & - b_{31}LP_d & & = A_3L \\
C_f \qquad\qquad & - b_{41}P_d & - b_{41}P_c & = A_4 \\
C_e \quad & + b_{51}P_d & & = A_5 \\
C_s & - b_{61}(P_d/I_d) & & = A_6 \\
C_h + C_f + C_e + C_s & & & = A_7 \\
b_{81}C_f \qquad\qquad & & + P_c & = A_8
\end{aligned}$$

When the first, third, and fourth equations are subtracted from the fifth equation and the terms are rearranged, the following equations are left:

$$\begin{aligned}
C_f \qquad & - b_{41}P_d & - b_{41}P_c & = A_4 \\
C_f + (b_{31}L - b_{51} + b_{61}/I_d)P_d & & & = A_7 - A_3L - A_5 - A_6 \\
b_{81}C_f \qquad & & + P_c & = A_8
\end{aligned}$$

If, for this reduced set of equations, the last equation is multiplied by $-b_{41}$ and subtracted from the first equation, the following system is given:

$$\begin{aligned}
(1 + b_{41}b_{81})C_f & - b_{41}P_d & = A_4 + b_{41}A_8 \\
C_f & + (b_{31}L - b_{51} + b_{61}/I_d)P_d & = A_7 - A_3L - A_5 - A_6
\end{aligned}$$

These 2 equations in 2 unknown can be solved easily by direct substitution or by inversion of the 2 by 2 matrix of coefficients of the endogenous variables. (See p. 26.)

COEFFICIENTS OF CORRELATION FOR SYSTEMS OF EQUATIONS

A number of correlation concepts can be considered in connection with systems of equations. Two sorts of correlation coefficients might be of interest: (1) Those that indicate the relative accuracy with which the individual endogenous variables can be estimated and (2) those that indicate the closeness of fit of the entire system. Methods for estimating each of these are discussed in this section.

Coefficients that are similar to multiple correlation coefficients or their squared values, commonly called coefficients of multiple determination, can be computed for the years included in the study by making use of the reduced form equations. (See p. 82.) For each year included in the analysis, an estimate of the particular endogenous variable is obtained and compared with the actual value for that year. If d equals the difference between these 2 values, the coefficient of determination for the variable i is given by:

$$R_i^2 = 1 - \frac{N\Sigma d^2}{m_{ii}}$$

The N is required in the numerator of the last term because of the use of augmented moments. Considerable computation may be required to obtain coefficients of this type as, for the wheat model for example, each of the reduced form equations involves 5 A's or (in effect) 14 predetermined variables. For computational purposes, it might be desirable in some models to write the reduced form equations in terms of the predetermined variables as such rather than in terms of the A's. For the wheat model, little would be gained by this procedure.

Measures of this sort, when multiplied by 100, show the percentage of variation in each endogenous variable explained by all of the predetermined variables in the system. In this sense, they are comparable to a multiple coefficient of determination. They probably would not, however, be subject to the same sampling distribution.

Theil (19, 20) has developed a statistic which he calls the coefficient of simultaneous correlation, S^2. This is analogous to a coefficient of multiple determination, R^2, in that it indicates the percentage of variation in the joint distribution of the endogenous variables included in the system explained by all of the predetermined variables in the system. 26/ Theil has shown that it can be consistently estimated from the sample. If in connection with fitting the equations the entire $M_{yz}M_{zz}^{-1}M_{zy}$ matrix was obtained, as for the lumber problem, this coefficient is easily obtained in the following way:

26/ This coefficient was called to the authors' attention by Roy Radner of the Cowles Foundation for Research in Economics. Theil suggests use of a similar coefficient for each structural equation. This shows the percentage of variation in the joint distribution of the endogenous variables in that equation explained by the predetermined variables in that equation. The authors of this handbook believe this to be of less interest than the two alternative coefficients discussed in the text.

$$S^2 = \frac{|M_{yz}M_{zz}^{-1}M_{zy}|}{|M_{yy}|}$$

These 2 determinants always are symmetrical. If of order greater than 3, they can be evaluated by the Doolittle method as shown in the appendix, p. 89. If of order $\leq$ 3, they can be evaluated by the methods discussed on p. 26. If, as for the wheat and dairy models, only part of the $M_{yz}M_{zz}^{-1}M_{zy}$ matrix was obtained, the rest of the matrix would need to be computed to obtain this coefficient.

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APPENDIX

Abbreviated Doolittle Method

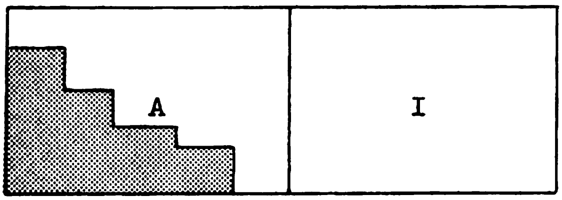
The forward solution of a method outlined by Doolittle (5) is illustrated in detail in tables 3 and 8. This scheme for inverting matrices and solving systems of equations is based upon methods developed earlier by Gauss (10). This approach, as it is usually explained, applies only to symmetrical matrices. In this section of the appendix, we apply it only to symmetrical matrices. In the next section, however, we show how a modification of the same method applies to the solution of any matrix, whether symmetrical or not.

Here we outline an abbreviated form of the forward solution of the Doolittle method. In abbreviated or compact methods we do as much work as possible on the calculating machine rather than on the worksheet. For example, in the method outlined here, the sum of two or more products is cumulated on the calculating machine; only the cumulated sum is copied. This eliminates the recording of many entries, and the possibility of making mistakes in the copying.

Outline 1 presents, in symbolic form, the abbreviated forward solution for a matrix of order n. <u>This method is applicable to symmetrical matrices only</u>. Methods for handling nonsymmetrical matrices are discussed on p. 95.

<u>Evaluation of Determinants</u>.--The abbreviated forward solution can be used to evaluate determinants. The value of a symmetrical determinant equals the product of the diagonal "α" elements of the abbreviated forward solution. Referring to outline 1, the value of the determinant of the matrix of order n is given by: $\alpha_{11}\ \alpha_{22}\ \alpha_{33} \cdots \alpha_{nn}$.

<u>Inversion of Matrices</u>.--To invert a symmetrical matrix, a worksheet is set up in the form:



where A is the matrix of order n to be inverted and I is the unit matrix of order n. An abbreviated forward solution is carried out on this augmented matrix, and the inverse is obtained by cumulating the products of certain elements from the I part of the forward solution as explained on p. 11.

Table 15 gives the abbreviated form of the computations involved in obtaining the inverse, or D, matrix shown in table 3. Note that the abbreviated form contains only the last two rows of each section of the forward solution of table 3. The abbreviated solution uses only one check column, that is, column (11), headed Σ. As many of the computations are carried directly in the machine, it is better practice to compute one check for the entire row of the forward

Outline 1.- Abbreviated forward solution

Computations

Row						
(1)	a_{11}	a_{12}	a_{13}	$\cdots$	a_{1n}	$a_{1\Sigma}$
(2)		a_{22}	a_{23}	$\cdots$	a_{2n}	$a_{2\Sigma}$
(3)			a_{33}	$\cdots$	a_{3n}	$a_{3\Sigma}$
$\cdot$				$\cdot$	$\cdot$	$\cdot$
$\cdot$				$\cdot$	$\cdot$	$\cdot$
$\cdot$				$\cdot$	$\cdot$	$\cdot$
(n)					a_{nn}	$a_{n\Sigma}$
(1)	α_{11}	α_{12}	α_{13}	$\cdots$	α_{1n}	$\alpha_{1\Sigma}$
(1")	1	β_{12}	β_{13}	$\cdots$	β_{1n}	$\beta_{1\Sigma}$
(2')		α_{22}	α_{23}	$\cdots$	α_{2n}	$\alpha_{2\Sigma}$
(2")		1	β_{23}	$\cdots$	β_{2n}	$\beta_{2\Sigma}$
(3')			α_{33}	$\cdots$	α_{3n}	$\alpha_{3\Sigma}$
(3")			1	$\cdots$	β_{3n}	$\beta_{3\Sigma}$
$\cdot$				$\cdot$	$\cdot$	$\cdot$
$\cdot$				$\cdot$	$\cdot$	$\cdot$
$\cdot$				$\cdot$	$\cdot$	$\cdot$
(n')					α_{nn}	$\alpha_{n\Sigma}$
(n")					1	$\beta_{n\Sigma}$

Instructions

Row

(1) - (n) Enter the matrix of order n.

The elements in the Σ column are obtained by summing across the row, including those terms omitted because of symmetry.

That is, $a_{i\Sigma} = \sum_{j=1}^{n} a_{ij}$ $(i=1,2,\ldots n)$

(1) $\alpha_{1j} = a_{1j}$ $(j=1,2,\ldots,n,\Sigma)$

Check: $\alpha_{1\Sigma} = \Sigma\alpha_{1j} (j=1, 2,\ldots n)$

(1") $\beta_{1j} = \alpha_{1j}/\alpha_{11} (j=2,\ldots,n,\Sigma)$

Check: $\beta_{1\Sigma} = \Sigma\beta_{1j} + 1$ $(j=2,\ldots n)$

(2') $\alpha_{2j} = a_{2j} - \beta_{12}\alpha_{1j}$ $(j=2,\ldots n,\Sigma)$

Check: $\alpha_{2\Sigma} = \Sigma\alpha_{2j}$ $(j=2,\ldots n)$

(2") $\beta_{2j} = \alpha_{2j}/\alpha_{22}$ $(j=3,\ldots n,\Sigma)$

Check: $\beta_{2\Sigma} = \Sigma\beta_{2j} + 1$ $(j=3,\ldots n)$

(3') $\alpha_{3j} = a_{3j} - \beta_{13}\,\alpha_{1j} - \beta_{23}\,\alpha_{2j}$
$(j=3,\ldots n,\Sigma)$

Check: $\alpha_{3\Sigma} = \Sigma\alpha_{3j}$ $(j=3,\ldots n)$

(3") $\beta_{3j} = \alpha_{3j}/\alpha_{33}$ $(j=4,\ldots n,\Sigma)$

Check: $\beta_{3\Sigma} = \Sigma\beta_{3j} + 1$ $(j=4,\ldots n)$

In general:

(k') $\alpha_{kj} = a_{kj} - \beta_{1k}\alpha_{1j} - \beta_{2k}\alpha_{2j} - \cdots - \beta_{k-1,k}\,\alpha_{k-1,j}$ $(j=k,\ldots n,\Sigma)$ Check: $\alpha_{k\Sigma} = \Sigma\alpha_{kj} (j=k,\ldots n)$

(k") $\beta_{kj} = \alpha_{kj}/\alpha_{kk}$ $(j=k+1,\ldots n,\Sigma)$ Check: $\beta_{k\Sigma} = \Sigma\beta_{kj} + 1$ $(j=k+1,\ldots n)$

Until: (n') $\alpha_{nj} = a_{nj} - \beta_{1n}\alpha_{1j} - \beta_{2n}\alpha_{2j} - \cdots - \beta_{n-1,n}\,\alpha_{n-1,j} (j=n,\Sigma)$ Check: $\alpha_{n\Sigma} = \alpha_{nn}$

(n") $\beta_{n\Sigma} = \alpha_{n\Sigma}/\alpha_{nn}$

Table 15.- Abbreviated forward solution and calculation of the inverse matrix for the 5-variable multiple regression problem 1/

Row	x_1 (1)	x_2 (2)	x_3 (3)	x_4 (4)	x_5 (5)	I_1 (6)	I_2 (7)	I_3 (8)	I_4 (9)	I_5 (10)	Σ (11)	Σ_I (12)
(1)	2.1088	2.2784	0.5799	0.6430	0.1419	1.	0	0	0	0	6.7521	
(2)		5.2090	.4121	2.1640	.3071	0	1.	0	0	0	11.3708	
(3)			.6735	.0940	.1065	0	0	1.	0	0	2.8663	
(4)				1.6118	-.0407	0	0	0	1.	0	5.4722	
(5)					.1063	0	0	0	0	1.	1.6212	
(1)	2.1088	2.2784	0.5799	0.6430	0.1419	1.	0	0	0	0	6.7521	1.
(1")	1.	1.0804	.2750	.3049	.0672	.4741	0	0	0	0	3.2018✓	.4741
(2')		2.7474	-.2144	1.4692	.1538	-1.0804	1.	0	0	0	4.0757✓	-.0804
(2")		1.	-.0780	.5347	.0559	-.3932	.3639	0	0	0	1.4834✓	-.0292
(3')			.4973	.0319	.0795	-.3593	.0780	1.	0	0	1.3275✓	.7187
(3")			1.	.0641	.1599	-.7225	.1569	2.0106	0	0	2.6692✓	1.4450
(4')				.6279	-.1714	.2958	-.5397	-.0641	1.	0	1.1484✓	.6919
(4")				1.	-.2729	.4711	-.8595	-.1021	1.5924	0	1.8288✓	1.1018
(5')					.0286	.1314	-.2158	-.1775	.2729	1.	1.0397✓	1.0111
(5")					1.	4.5847	-7.5270	-6.1904	9.5205	34.8749	36.2627✓	35.2627

Calculation of the inverse, or D, matrix

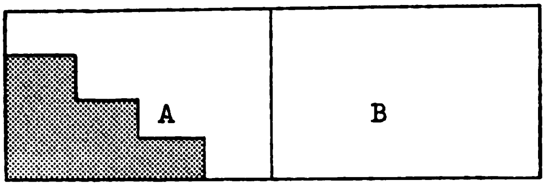
	(1)	(2)	(3)	(4)	(5)	(6)
(1)	1.9008	-1.6934	-1.5665	1.7227	4.5847	4.9482✓
(2)		2.4647	1.5481	-2.9143	-7.5270	-8.1220✓
(3)			3.1160	-1.7920	-6.1904	-4.8848✓
(4)				4.1914	9.5205	10.7282✓
(5)					34.8749	35.2627✓

1/ These computations were performed with 9 decimal places, of which only 4 appear in the table; therefore, some of the computations may appear slightly in error.

solution than to make separate checks for each part of the row. The elements in column (12), headed Σ_I, are obtained by summing across only the I columns of the rows. These elements are obtained after the forward solution has been completed and are used to check the computation of the inverse matrix.

The computation of the inverse, or D, matrix from the abbreviated forward solution is the same as that of the full forward solution, as explained on p. 11. The formula for d_{ij}, the ijth element of the matrix D of order 5 given on p. 11, can easily be extended to apply to a matrix of any order.

Computation of the Matrix Product $B'A^{-1}B$.--The abbreviated forward solution is also used to obtain the matrix product $B'A^{-1}B$. In this case, B need not be symmetrical. In any case, it is written out in full. The worksheet is set up in the form:



An abbreviated forward solution is carried out on this augmented matrix and $B'A^{-1}B$ is obtained by cumulating the products of certain elements from the B part of the forward solution. This computation is analogous to the computation of the inverse matrix, with matrix B replacing unit matrix I.

Table 16 gives the abbreviated form of the forward solution for the computation where $A = M_{zz}$ and $B = M_{zy}$ shown in table 8. Note that the abbreviated form makes use of only one check column, Σ.

Computation of the Matrix Product $A^{-1}B$.--The matrix product $A^{-1}B$ can be efficiently computed with an abbreviated forward and back solution. Table 17 gives the abbreviated forward and back solution for the computation of the transpose of $A^{-1}B$, where $A = B_{y*y*}$ and $B = W_{y*y*}$, which is shown in complete detail in table 12. Again B is written out in full.

Outline 2 presents, in symbolic form, the computation of the transpose of $A^{-1}B$, where A is the symmetrical n x n matrix shown in rows (1) through (n) and columns (1) through (n), and B is the n x m matrix, which need not be symmetrical, shown in rows (1) through (n) and columns (n+1) through (n+m). A forward solution is carried out and the transpose of $A^{-1}B$ is obtained in the form of a back solution beginning with the last column, that is, γ'_{in}; continuing to the next to the last column, $\gamma'_{i,n-1}$; and so forth, until the first column, γ'_{i1}, is obtained. Table 17 illustrates the computations for a problem in which $n=m=2$.

If $m = n$ and B is a unit matrix of order n, this method can be used to compute A^{-1}. This method actually gives $(A^{-1})'$, but since A is symmetrical, A^{-1} is also symmetrical, and $(A^{-1})' = A^{-1}$.

Table 16.- Abbreviated forward solution for data shown in table 8 1/

Row	M_{zz}			M_{zy}		Σ	Σ_y
	z_1	z_2	z_3	y_1	y_2		
(1) z_1	0.5127	0.5017	0.4060	0.6138	0.4940	2.5284	
(2) z_2		.5543	-.0056	.4456	.5048	2.0008	
(3) z_3			9.3676	1.1593	-1.2651	9.6622	
(1)	0.5127	0.5017	0.4060	0.6138	0.4940	2.5284	1.1078
(1")	1.	.9785	.7918	1.1970	.9634	4.9308✓	2.1604
(2')		.0633	-.4030	-.1549	.0214	-.4732✓	-.1335
(2")		1.	-6.3653	-2.4481	.3383	-7.4751✓	-2.1098
(3')			6.4807	-.3132	-1.5200	4.6474✓	-1.8332
(3")			1.	-.0483	-.2345	.7171✓	-.2828

Computation of $M_{yz}M_{zz}^{-1}M_{zy}$

	y_1	y_2	Σ
y_1	1.1293	0.6124	1.7417✓
y_2		.8397	1.4521✓

$M_{y*z}M_{zz}^{-1}M_{zy*}$ for equation (2.1)

	y_2	y_1	Σ
y_2	0.8397	0.6124	1.4521✓
y_1		1.1293	1.7417✓

1/ These computations were performed with 9 decimal places of which only 4 appear in the table; therefore, some of the computations may appear slightly in error.

Table 17.- Abbreviated form for computations shown in table 12 1/

Section (9.1) - Forward solution

Row	B_{y*y*}		W_{y*y*}		Σ
	y_2	y_1	y_2	y_1	
y_2	0.6688	0.7689	0.3183	0.2550	2.0112
y_1		.9858	.2550	1.0930	3.1028
(1)	0.6688	0.7689	0.3183	0.2550	2.0112
(1")	1.	1.1496	.4759	.3812	3.0069✓
(2')		.1017	-.1110	.7998	.7905✓
(2")		1.	-1.0912	7.8617	7.7704✓

Section (10.1) - Back solution for $A' = B^{-1}W$

	1.7305	-1.0912
	-8.6573	7.8617
	-5.9267✓	7.7704✓

1/ These computations were performed with 9 decimal places of which 4 appear in the table; therefore, some of the computations may appear to be slightly in error.

Outline 2.- Abbreviated forward and back solution

Computations

$$a_{11}\,a_{12}\,a_{13}\cdots a_{1n}\,a_{1,n+1}\,a_{1,n+2}\cdots a_{1,n+m}\,a_{1\Sigma}$$
$$a_{22}\,a_{23}\cdots a_{2n}\,a_{2,n+1}\,a_{2,n+2}\cdots a_{2,n+m}\,a_{2\Sigma}$$
$$a_{33}\cdots a_{3n}\,a_{3,n+1}\,a_{3,n+2}\cdots a_{3,n+m}\,a_{3\Sigma}$$
$$\vdots$$
$$a_{nn}\,a_{n,n+1}\,a_{n,n+2}\cdots a_{n,n+m}\,a_{n\Sigma}$$

Instructions

Carry out a forward solution as indicated in outline 1.

In general:

$$\alpha_{kj}=a_{kj}-\beta_{1k}\alpha_{1j}-\beta_{2k}\alpha_{2j}-\cdots-\beta_{k-1,k}\alpha_{k-1,j}$$
$$(j=k,\ldots n,n+1,\ldots n+m,\Sigma)$$

Check: $\alpha_{k\Sigma} = \Sigma\alpha_{kj}$ $(j=k,\ldots n,\ n+1,\ldots n+m)$

$$\beta_{kj}=\alpha_{kj}/\alpha_{kk}\ (j=k+1,\ldots n,n+1,\ldots n+m,\Sigma)$$

Check: $\beta_{k\Sigma}=\Sigma\beta_{kj}+1$ $(j=k+1,\ldots n,n+1,\ldots n+m)$

Forward solution

$$\alpha_{11}\,\alpha_{12}\,\alpha_{13}\cdots\alpha_{1n}\,\alpha_{1,n+1}\,\alpha_{1,n+2}\cdots\alpha_{1,n+m}\,\alpha_{1\Sigma}$$
$$1\quad\beta_{12}\,\beta_{13}\cdots\beta_{1n}\,\beta_{1,n+1}\,\beta_{1,n+2}\cdots\beta_{1,n+m}\,\beta_{1\Sigma}$$
$$\alpha_{22}\,\alpha_{23}\cdots\alpha_{2n}\,\alpha_{2,n+1}\,\alpha_{2,n+2}\cdots\alpha_{2,n+m}\,\alpha_{2\Sigma}$$
$$1\quad\beta_{23}\cdots\beta_{2n}\,\beta_{2,n+1}\,\beta_{2,n+2}\cdots\beta_{2,n+m}\,\beta_{2\Sigma}$$
$$\vdots$$
$$\alpha_{nn}\,\alpha_{n,n+1}\,\alpha_{n,n+2}\cdots\alpha_{n,n+m}\,\alpha_{n\Sigma}$$
$$1\quad\beta_{n,n+1}\,\beta_{n,n+2}\cdots\beta_{n,n+m}\,\beta_{n\Sigma}$$

Back solution

(1) $\gamma'_{in} = \beta_{n,n+i}$ $(i=1,\ldots m,\Sigma)$

Check: $\gamma'_{\Sigma n} = \Sigma\gamma'_{in} + 1$ $(i=1,\ldots m)$

(2) $\gamma'_{i,\,n-1} = \beta_{n-1,n+i} - \gamma'_{in}\beta_{n-1,n}$ $(i=1,\ldots m,\Sigma)$

Check: $\gamma'_{\Sigma,n-1} = \Sigma\gamma'_{i,n-1} + 1$ $(i=1,\ldots m)$

(3) $\gamma'_{i,n-2}=\beta_{n-2,n+i}-\gamma'_{in}\beta_{n-2,n}-\gamma'_{i,n-1}\beta_{n-2,n-1}$
$(i=1,\ldots m,\Sigma)$

Check: $\gamma'_{\Sigma,n-2}=\Sigma\gamma'_{i,n-2} + 1$ $(i=1,\ldots m)$

In general:

$$\gamma'_{ik} = \beta_{k,n+i}-\gamma'_{in}\beta_{kn}-\gamma'_{i,n-1}\beta_{k,n-1}-\cdots$$
$$-\gamma'_{i,k+1}\beta_{k,k+1}\ (i=1,\ldots m,\Sigma)$$

Check: $\gamma'_{\Sigma k} = \Sigma\gamma'_{ik} + 1$ $(i=1,\ldots m)$

Until:

$$\gamma'_{i1}=\beta_{1,n+i}-\gamma'_{in}\beta_{1n}-\gamma'_{i,n-1}\beta_{1,n-1}-\cdots$$
$$-\gamma'_{i2}\beta_{12}\quad(i=1,\ldots m,\Sigma)$$

Check: $\gamma'_{\Sigma 1} = \Sigma\gamma'_{i1} + 1$ $(i=1,\ldots m)$

Back solution

$$\gamma'_{11}\cdots\gamma'_{1,n-2}\,\gamma'_{1,n-1}\,\gamma'_{1n}$$
$$\gamma'_{21}\cdots\gamma'_{2,n-2}\,\gamma'_{2,n-1}\,\gamma'_{2n}$$
$$\gamma'_{31}\cdots\gamma'_{3,n-2}\,\gamma'_{3,n-1}\,\gamma'_{3n}$$
$$\vdots$$
$$\gamma'_{m1}\cdots\gamma'_{m,n-2}\,\gamma'_{m,n-1}\,\gamma'_{mn}$$
$$\gamma'_{\Sigma 1}\cdots\gamma'_{\Sigma,n-2}\,\gamma'_{\Sigma,n-1}\,\gamma'_{\Sigma n}$$

(3) (2) (1)

If $m = 1$, this method can be used to solve the system of equations:

$$a_{11}X_1 + a_{12}X_2 + \cdot \quad \cdot \quad \cdot + a_{1n}X_n = a_{1,n+1}$$

$$a_{12}X_1 + a_{22}X_2 + \cdot \quad \cdot \quad \cdot + a_{2n}X_n = a_{2,n+1}$$

$$\vdots$$

$$a_{1n}X_1 + a_{2n}X_2 + \cdot \quad \cdot \quad \cdot + a_{nn}X_n = a_{n,n+1}$$

The back solution provides the solution in the form:

$$\gamma'_{ij} = X_j \qquad (j=1, \ldots n)$$

This eliminates one step in the usual matrix solution of a system of simultaneous equations-- multiplication of the inverse of the matrix of the coefficients by the vector of the numbers to the right of the equality sign.

An efficient method for computing the inverse matrix without a back solution and without the reduction of the unit matrix is given by Waugh (22). This approach is not included in this handbook because it does not relate directly to any specific step in the statistical fitting of systems of simultaneous equations.

A General Method

The abbreviated Doolittle method as we have described it is applicable only when A is symmetrical. But a slight modification of the method makes it applicable to any matrix, whether symmetrical or not.

The method outlined below is commonly known as the Crout method (4). Similar methods have been developed by Waugh and Dwyer (23). The methods of Crout and Waugh and Dwyer require no more work than the usual Doolittle method. It is unnecessary to learn two different methods--one for symmetrical matrices and one for nonsymmetrical matrices. The statistician or statistical clerk who becomes familiar with the method outlined in the following pages will be able to solve any set of equations and invert any matrix.

In this handbook, however, this method is used only for nonsymmetrical matrices. This was done chiefly because many statistical clerks and research workers are familiar with the Doolittle method as applied to symmetrical matrices. They will not have to learn a new approach for many of the operations we have described.

Solving Systems of Equations.--The system of equations:

$$\left.\begin{aligned} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n &= a_{10} \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n &= a_{20} \\ &\vdots \\ a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{nn}X_n &= a_{n0} \end{aligned}\right\} \quad (1)$$

where $a_{ij} \neq a_{ji}$ cannot be solved by the methods discussed up to this point.

If $n = 2$, the equations can be solved easily by direct substitution. If $n = 3$, the above equations can be efficiently solved by Cramer's Rule, which says that:

$$X_1 = \frac{\begin{vmatrix} a_{10} & a_{12} & a_{13} \\ a_{20} & a_{22} & a_{23} \\ a_{30} & a_{32} & a_{33} \end{vmatrix}}{\Delta} \qquad X_2 = \frac{\begin{vmatrix} a_{11} & a_{10} & a_{13} \\ a_{21} & a_{20} & a_{23} \\ a_{31} & a_{30} & a_{33} \end{vmatrix}}{\Delta}$$

$$X_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & a_{10} \\ a_{21} & a_{22} & a_{20} \\ a_{31} & a_{32} & a_{30} \end{vmatrix}}{\Delta} \qquad \text{where } \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The value of a third order determinant is given on p. 26. When $n > 3$, this method is not efficient, since the evaluation of higher order determinants is cumbersome. Therefore when $n > 3$, we use the Crout method to solve the system of equations.

Outline 3 presents, in symbolic form, the Crout method for solving a system of equations. An illustration of this method to solve the following equations is given in table 18.

$$\left.\begin{aligned} C_f - 2.5P_c + 2.5P_d &= 203.9 \\ 0.0489C_f + P_c &= 138.1 \\ C_f - 10.467P_d &= -1866.79 \end{aligned}\right\} \quad (2)$$

These equations could be solved easily by Cramer's Rule or by the approach described on p. 85; the Crout method is shown for illustrative purposes.

Outline 3.- Crout method for solving a system of linear equations

Computations

Given the system of equations in n unknowns

$$a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + \dots + a_{1n}X_n = a_{10}$$
$$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + \dots + a_{2n}X_n = a_{20}$$
$$a_{31}X_1 + a_{32}X_2 + a_{33}X_3 + \dots + a_{3n}X_n = a_{30}$$
$$\vdots$$
$$a_{n1}X_1 + a_{n2}X_2 + a_{n3}X_3 + \dots + a_{nn}X_n = a_{n0}$$

Given matrix

$$\begin{array}{cccccc}
a_{11} & a_{12} & a_{13} \cdots & a_{1n} & a_{10} & a_{1\Sigma} \\
a_{21} & a_{22} & a_{23} \cdots & a_{2n} & a_{20} & a_{2\Sigma} \\
a_{31} & a_{32} & a_{33} \cdots & a_{3n} & a_{30} & a_{3\Sigma} \\
\vdots & & & & & \\
a_{n1} & a_{n2} & a_{n3} \cdots & a_{nn} & a_{n0} & a_{n\Sigma}
\end{array}$$

Auxiliary matrix

$$\begin{array}{c|c|c|c|cccl}
b_{11} & b_{12} & b_{13} & \cdots & b_{1n} & b_{10} & b_{1\Sigma} & (2) \\
b_{21} & b_{22} & b_{23} & \cdots & b_{2n} & b_{20} & b_{2\Sigma} & (4) \\
b_{31} & b_{32} & b_{33} & \cdots & b_{3n} & b_{30} & b_{3\Sigma} & (6) \\
\vdots & & & & & & & \\
b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} & b_{n0} & b_{n\Sigma} & (2n) \\
(1) & (3) & (5) & & (2n-1) & & &
\end{array}$$

Solution

$$\begin{array}{ccccc}
X_1 & \cdots & X_{n-2} & X_{n-1} & X_n \\
X^*_1 & \cdots & X^*_{n-2} & X^*_{n-1} & X^*_n \\
 & & (3) & (2) & (1)
\end{array}$$

Instructions

Given matrix

$a_{i\Sigma} = \sum_{j=1}^{n} a_{ij}$ $(i=1,2,\dots n,0)$ That is, the sum across the ith row.

Auxiliary matrix

(1) $b_{i1} = a_{i1}$ $(i=1,\dots n)$

(2) $b_{1j} = a_{1j}/a_{11}$ $(j=2,\dots n,0,\Sigma)$

Check: $b_{1\Sigma} = \Sigma b_{1j} + 1$ $(j=2,\dots n,0)$

(3) $b_{22} = a_{22} - b_{21}b_{12}$

$b_{i2} = a_{i2} - b_{i1}b_{12}$ $(i=3,\dots n)$

(4) $b_{2j} = (a_{2j} - b_{21}b_{1j})/b_{22}$ $(j=3,\dots n,0,\Sigma)$

Check: $b_{2\Sigma} = \Sigma b_{2j} + 1$ $(j=3,\dots n,0)$

(5) $b_{33} = a_{33} - b_{31}b_{13} - b_{32}b_{23}$

$b_{i3} = a_{i3} - b_{i1}b_{13} - b_{i2}b_{23}$ $(i=4,\dots n)$

(6) $b_{3j} = (a_{3j} - b_{31}b_{1j} - b_{32}b_{2j})/b_{33}$ $(j=4,\dots n,0,\Sigma)$

Check: $b_{3\Sigma} = \Sigma b_{3j} + 1$ $(j=4,\dots n,0)$

In general:

$b_{kk} = a_{kk} - b_{k1}b_{1k} - b_{k2}b_{2k} - b_{k3}b_{3k} - \dots - b_{k,k-1}b_{k-1,k}$

$b_{ik} = a_{ik} - b_{i1}b_{1k} - b_{i2}b_{2k} - b_{i3}b_{3k} - \dots - b_{i,k-1}b_{k-1,k}$ $(i=k+1,\dots n)$

$b_{kj} = (a_{kj} - b_{k1}b_{1j} - b_{k2}b_{2j} - b_{k3}b_{3j} - \dots - b_{k,k-1}b_{k-1,j})/b_{kk}$

$(j=k+1,\dots n,0,\Sigma)$ Check: $b_{k\Sigma} = \Sigma b_{kj} + 1$ $(j=k+1,\dots n,0)$

Until:

(2n-1) $b_{nn} = a_{nn} - b_{n1}b_{1n} - b_{n2}b_{2n} - \dots - b_{n,n-1}b_{n-1,n}$

(2n) $b_{nj} = (a_{nj} - b_{n1}b_{1j} - b_{n2}b_{2j} - \dots - b_{n,n-1}b_{n-1,j})/b_{nn}$ $(j=0,\Sigma)$

Check: $b_{n\Sigma} = b_{n0} + 1$

Solution

(1) $X_n = b_{n0}$ $X^*_n = b_{n\Sigma}$ Check: $X^*_n = X_n + 1$

(2) $X_{n-1} = b_{n-1,0} - b_{n-1,n}X_n$ $X^*_{n-1} = b_{n-1,\Sigma} - b_{n-1,n}X^*_n$

Check: $X^*_{n-1} = X_{n-1} + 1$

(3) $X_{n-2} = b_{n-2,0} - b_{n-2,n}X_n - b_{n-2,n-1}X_{n-1}$

$X^*_{n-2} = b_{n-2,\Sigma} - b_{n-2,n}X^*_n - b_{n-2,n-1}X^*_{n-1}$ Check: $X^*_{n-2} = X_{n-2} + 1$

In general:

$X_k = b_{k0} - b_{kn}X_n - b_{k,n-1}X_{n-1} - b_{k,n-2}X_{n-2} - \dots - b_{k,k+1}X_{k+1}$

$X^*_k = b_{k\Sigma} - b_{kn}X^*_n - b_{k,n-1}X^*_{n-1} - b_{k,n-2}X^*_{n-2} - \dots - b_{k,k+1}X^*_{k+1}$

Check: $X^*_k = X_k + 1$

The final check is the substitution of the unknowns in the original equation.

Table 18.- Crout method for solving a system of equations 1/

Coefficient for -			Constant term	Σ
C_f	P_c	P_d		
Given matrix:				
1.	-2.5000	2.5000	203.9000	204.9000
.0489	1.	0	138.1000	139.1489
1.	0	-10.4670	-1866.7900	-1876.2570
Auxiliary matrix:				
1.0000	-2.5000	2.5000	203.9000	204.9000 ✓
.0489	1.1222	-.1089	114.1717	115.0628 ✓
1.0000	2.5000	-12.6946	185.5991	186.5991 ✓
Solution:				Final check:
				203.9000 ✓
75.8762	134.3896	185.5991		138.0999 ✓
76.8762 ✓	135.3896 ✓	186.5991 ✓		-1866.7899 ✓

1/ These computations were performed with 9 decimal places, of which only 4 appear in the table; therefore some of the computations may appear slightly in error.

Matrix Inversion.--The Crout method also can be used to invert nonsymmetrical matrices. The computations involved in inverting a matrix of order n are indicated in outline 4. The matrix to be inverted is given in rows (1) through (n), columns (1) through (n), and the unit matrix is written in rows (1) through (n), columns (n+1) through (n+n). The transpose of the inverse matrix is obtained in the form of a back solution, with the last column, that is, e_{in}, computed first.

The system of equations (1) can be written, in matrix notation, in the form:

$$AX' = a'$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad X' = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \quad \text{and } a' = \begin{bmatrix} a_{10} \\ a_{20} \\ \vdots \\ a_{n0} \end{bmatrix}$$

Outline 4.- Crout method for matrix inversion

Computations

Given matrix

$a_{11}a_{12}a_{13}\cdots a_{1n}a_{1,n+1}a_{1,n+2}\cdots a_{1,n+n}a_{1\Sigma}$

$a_{21}a_{22}a_{23}\cdots a_{2n}a_{2,n+1}a_{2,n+2}\cdots a_{2,n+n}a_{2\Sigma}$

$a_{31}a_{32}a_{33}\cdots a_{3n}a_{3,n+1}a_{3,n+2}\cdots a_{3,n+n}a_{3\Sigma}$

.
.
.

$a_{n1}a_{n2}a_{n3}\cdots a_{nn}a_{n,n+1}a_{n,n+2}\cdots a_{n,n+n}a_{n\Sigma}$

Auxiliary matrix

$b_{11}b_{12}b_{13}\cdots b_{1n}b_{1,n+1}b_{1,n+2}\cdots b_{1,n+n}b_{1\Sigma}$

$b_{21}b_{22}b_{23}\cdots b_{2n}b_{2,n+1}b_{2,n+2}\cdots b_{2,n+n}b_{2\Sigma}$

$b_{31}b_{32}b_{33}\cdots b_{3n}b_{3,n+1}b_{3,n+2}\cdots b_{3,n+n}b_{3\Sigma}$

.
.
.

$b_{n1}b_{n2}b_{n3}\cdots b_{nn}b_{n,n+1}b_{n,n+2}\cdots b_{n,n+n}b_{n\Sigma}$

Solution

$e_{11}\ \cdots\ e_{1,n-2}\ e_{1,n-1}\ e_{1n}$

$e_{21}\ \cdots\ e_{2,n-2}\ e_{2,n-1}\ e_{2n}$

$e_{31}\ \cdots\ e_{3,n-2}\ e_{3,n-1}\ e_{2n}$

.
.
.

$e_{n1}\ \cdots\ e_{n,n-2}\ e_{n,n-1}\ e_{nn}$

$e_{\Sigma 1}\ \cdots\ e_{\Sigma,n-2}\ e_{\Sigma,n-1}\ e_{\Sigma n}$

(3) (2) (1)

Instructions

Given matrix

$\begin{bmatrix} a_{ij} \end{bmatrix}$ $(i,j = 1, \ldots n)$ is the matrix to be inverted.

$\begin{bmatrix} a_{ij} \end{bmatrix}$ $(i=1, \ldots n;\ j=n+1,\ldots n+n)$ is the unit matrix.

$$a_{i\Sigma} = \sum_{j=1}^{n} a_{ij} \quad (i=1,\ldots n,\ n+1,\ldots n+n)$$

that is, the sum across the ith row.

Auxiliary matrix

The auxiliary matrix is computed in the manner indicated in outline 3.

Solution

(1) $e_{in} = b_{n,n+i}$ $(i=1, \ldots n,\Sigma)$

Check: $e_{\Sigma n} = \Sigma e_{in} + 1$ $(i=1, \ldots n)$

(2) $e_{i,n-1} = b_{n-1,n+i} - e_{in}b_{n-1,n}$ $(i=1, \ldots n,\Sigma)$

Check: $e_{\Sigma,n-1} = \Sigma e_{i,n-1} + 1$ $(i=1, \ldots n)$

(3) $e_{i,n-2} = b_{n-2,n+1} - e_{in}b_{n-2,n} - e_{i,n-1}b_{n-2,n-1}$ $(i=1,\ldots,n,\Sigma)$

Check: $e_{\Sigma,n-2} = \Sigma e_{i,n-2} + 1$ $(i=1,\ldots n)$

In general: $e_{ik} = b_{k,n+i} - e_{in}b_{kn} - e_{i,n-1}b_{k,n-1} - \cdots - e_{i,k+1}b_{k,k+1}$

$(i=1,\ldots n,\Sigma)$ Check: $e_{\Sigma k} = \Sigma e_{ik} + 1$ $(i=1,\ldots n)$

As shown on p. 82, a solution to the equations can be found by premultiplying a' by A^{-1}.

In using a system of simultaneous equations for analytical purposes, the matrix A usually consists of a set of constants, but a' varies from period to period (see p. 82). In such cases, it is convenient to compute A^{-1} and then postmultiply it by the varying a' to get the X' values for each period. Equations (2) are solved by this method in table 19.

For this example,

$$A = \begin{bmatrix} 1 & -2.5 & 2.5 \\ .0489 & 1 & 0 \\ 1 & 0 & -10.467 \end{bmatrix} \quad X' = \begin{bmatrix} C_f \\ P_c \\ P_d \end{bmatrix} \text{ and } a' = \begin{bmatrix} 203.9 \\ 138.1 \\ -1866.79 \end{bmatrix}$$

Since the Crout method gives $[A^{-1}]'$, X' is obtained by a column-by-column multiplication of $[A^{-1}]'$ by a'. See p.101 for an explanation of this and p. 103 for the check on the computation.

If only one or two elements of A are subject to change, as with nonlinear variables, a new auxiliary matrix need not be computed. Only those terms of the auxiliary matrix that are affected by the changing a_{ij} values need be computed. These terms can be determined from outline 3 or 4.

Table 19.- Crout method for solving a system of equations using the inverse matrix 1/

C_f	P_c	P_d	I_1	I_2	I_3	Σ
Given matrix:						
1.0000	-2.5000	2.5000	1	0	0	2.0000
.0489	1.0000	.0000	0	1	0	2.0489
1.0000	.0000	-10.4670	0	0	1	-8.4670
Auxiliary matrix:						
1.0000	-2.5000	2.5000	1	0	0	2.0000 ✓
.0489	1.1222	-.1089	-.0435	.8910	0	1.7385 ✓
1.0000	2.5000	-12.6946	.0701	.1754	-.0787	1.1668 ✓
Transpose of the inverse matrix:						Solution:
			Σ			
0.7347	-0.0359	0.0701	0.7689		203.9	75.8762
1.8367	.9101	.1754	2.9224		138.1	134.3896
.1754	-.0085	-.0787	.0881		-1866.79	185.5991
3.7469 ✓	1.8656 ✓	1.1668 ✓				Σ' 395.8650 ✓

1/ These computations were performed with 9 decimal places, of which only 4 appear in the table; therefore some of the computations may appear slightly in error.

Evaluation of Determinants.--The Crout method also can be used to evaluate nonsymmetrical determinants. Referring to outline 3, the value of the determinant $|a_{ij}|$ $(i,j=1, \ldots n)$ is given by $b_{11}\, b_{22}\, b_{33} \ldots b_{nn}$.

Additional Comments on Matrix Multiplication

Alternative Forms.--Matrix multiplication is defined on p.24 in terms of a row-by-column operation; that is, e_{ij}, the ij th element of the matrix product, E = AB, equals the sum of the products of the elements in the ith row of A with the elements in the jth column of B, starting at the left and the top respectively.

Matrix multiplication also can be defined as a row-by-row operation, that is, e_{ij}, the ij th element in the matrix product, E = AB, equals the sum of the products of the elements in the ith row of A with the elements in the jth row of B' (the transpose of B). Or it can be defined as a column-by-column operation where e_{ij}, the ijth element in the matrix product, E = AB, equals the sum of the products of the elements of the ith column of A' with the elements in the jth column of B. If A and B are symmetrical, the matrix product AB can be obtained by either (1) a row-by-column, (2) a row-by-row, or (3) a column-by-column multiplication of A with B.

Row-by-row or column-by-column multiplications are useful concepts, especially when either B' or A' has been computed and the matrix product AB is needed. It saves copying data and its attendent mistakes. Referring back to the limited information computations, in section (7) $M_{y*z*}P'$ is obtained by a column-by-column multiplication of $(M_{y*z*})' = M_{z*y*}$ with P' (see p. 37). In section (26), $U = {}_{01}P'F_{bc}$ is obtained by a row-by-row multiplication of ${}_{01}P'$ with F'_{bc} (see p.47). bW could have been computed in section (16) by a column-by-column multiplication of b' with W (see p.44). As a means of separating the computation of the coefficients and their standard errors, a row-by-column multiplication of b with W is shown instead in tables 9 and 10.

Checks.--In the explanation of the limited information computations, checks on the computation of the matrix product AB are introduced for certain specified forms of A and B. These include: (1) Augmenting B with a Σ column, composed of row sums. After the multiplication is carried out on the augmented B matrix, the ith element of the resulting Σ column of AB is checked against the sum across the ith row of AB. See, for example, the computation of M^{-1}_{z*z*} times $M_{z*y*} = P'$ in table 10. (2) Augmenting A with a row, composed of column sums. After the multiplication is carried out on the augmented A matrix, the jth element of the resulting Σ' row of AB is checked against the sum down the jth column of AB. See for example, the computation of -P' times b ' = c' in table 10. (3) A recomputation where the product is a scalar. See, for example, the computation of -P' times b ' = c' in table 9.

The same general procedure, that is, the use of a Σ column or a Σ' row, can be extended so that a check is available for the computation of any matrix product obtained by any method of matrix multiplication. These results are summarized in outline 5.

This outline is divided into three sections: Computations that relate to (a) row-by-column multiplication; (b) row-by-row multiplication; and (c) column-by-column multiplication. Under each of these methods are listed all possible

Outline 5.- Computational forms for checking matrix multiplications

Row-by-column multiplication

Type of multiplication				Computational form				
Matrix x matrix	=	matrix	or	$[\ ]$	x	$[\ ]^{\Sigma}$	=	$[\ ]^{\Sigma}$
				${}_{\Sigma'}[\ ]$	x	$[\ ]$	=	${}_{\Sigma'}[\ ]$
Column x row	=	matrix	or	$[\,]$	x	$[\ ]^{\Sigma}$	=	$[\ ]^{\Sigma}$
				${}_{\Sigma'}[\,]$	x	$[\ ]$	=	${}_{\Sigma'}[\ ]$
Row x matrix	=	row		$[\ ]$	x	$[\ ]^{\Sigma}$	=	$[\ ]^{\Sigma}$
Scalar x row	=	row		$[\,]$	x	$[\ ]^{\Sigma}$	=	$[\ ]^{\Sigma}$
Matrix x column	=	column		${}_{\Sigma'}[\ ]$	x	$[\,]$	=	${}_{\Sigma'}[\,]$
Column x scalar	=	column		${}_{\Sigma'}[\,]$	x	$[\,]$	=	${}_{\Sigma'}[\,]$
Row x column	=	scalar		Scalar product-recompute				
Scalar x scalar	=	scalar		Do.				

Row-by-row multiplication

Matrix x matrix	=	matrix	or	$[\ ]$	x	${}_{\Sigma'}[\ ]$	=	$[\ ]^{\Sigma}$
				${}_{\Sigma'}[\ ]$	x	$[\ ]$	=	${}_{\Sigma'}[\ ]$
Column x column	=	matrix	or	$[\,]$	x	${}_{\Sigma'}[\,]$	=	$[\ ]^{\Sigma}$
				${}_{\Sigma'}[\,]$	x	$[\,]$	=	${}_{\Sigma'}[\ ]$
Row x matrix	=	row		$[\ ]$	x	${}_{\Sigma'}[\ ]$	=	$[\ ]^{\Sigma}$
Scalar x column	=	row		$[\,]$	x	${}_{\Sigma'}[\,]$	=	$[\ ]^{\Sigma}$
Matrix x row	=	column		${}_{\Sigma'}[\ ]$	x	$[\ ]$	=	${}_{\Sigma'}[\,]$
Column x scalar	=	column		${}_{\Sigma'}[\,]$	x	$[\,]$	=	${}_{\Sigma'}[\,]$
Row x row	=	scalar		Scalar product-recompute				
Scalar x scalar	=	scalar		Do.				

Column-by-column multiplication

Matrix x matrix	=	matrix	or	$[\ ]$	x	$[\ ]^{\Sigma}$	=	$[\ ]^{\Sigma}$
				$[\ ]^{\Sigma}$	x	$[\ ]$	=	${}_{\Sigma'}[\ ]$
Row x row	=	matrix	or	$[\ ]$	x	$[\ ]^{\Sigma}$	=	$[\ ]^{\Sigma}$
				$[\ ]^{\Sigma}$	x	$[\ ]$	=	${}_{\Sigma'}[\ ]$
Column x matrix	=	row		$[\,]$	x	$[\ ]^{\Sigma}$	=	$[\ ]^{\Sigma}$
Scalar x row	=	row		$[\,]$	x	$[\ ]^{\Sigma}$	=	$[\ ]^{\Sigma}$
Matrix x column	=	column		$[\ ]^{\Sigma}$	x	$[\,]$	=	${}_{\Sigma'}[\,]$
Row x scalar	=	column		$[\ ]^{\Sigma}$	x	$[\,]$	=	${}_{\Sigma'}[\,]$
Column x column	=	scalar		Scalar product-recompute				
Scalar x scalar	=	scalar		Do.				

types of multiplication. The word "column" refers to a column vector and the word "row," to a row vector. Next to each of these is the computational form that should be used to check the operation. The symbol [   ] denotes a matrix; [ ] a column vector; [   ] a row vector; [ ] a scalar; Σ a column of row sums; and Σ' a row of column sums.

As an illustration of the use of outline 5, consider the following: a column-by-column multiplication of a matrix times a column vector. (This method was used on p. 71.) Outline 5 indicates that we augment the matrix with a Σ column. After the column-by-column multiplication is carried out (as explained on p. 101), the resulting column vector has an additional row, Σ'. That the sum of the elements of the column vector is equal to the element in the Σ' row is indicated by a check mark. This checks the computation.

For several types of matrix multiplication, alternate computational forms are indicated in outline 5. The preferable form to use is determined by: (a) The form of the matrices used to compute the product, that is, whether a Σ column or Σ' row has already been computed; or (b) further computations, if any, into which the product enters, that is whether a Σ column of Σ' row is desired.

GLOSSARY 27/

Adjoint of a matrix.--In this handbook this is used as a computational device for inverting a (2x2) matrix (see p. 40). For a general definition, see Klein (13, pp. 335-337) or other references on matrices.

Adjusted moments.--Moments that have been adjusted in such a way as to make the augmented sums of squares for each variable nearly equal to 1.

Adjustment factors.--Factors used to obtain the adjusted moments from the augmented moments (see p. 6).

Augmented matrix.--Two or more matrices or vectors written adjacently to facilitate computational operations. The reader is cautioned not to confuse this with an augmented moment matrix.

Augmented moment matrix.--A matrix consisting of adjusted augmented moments (see p. 7).

Augmented moments.--Moments are sums of squares and cross-products from which a correction factor has been deducted to give results that would have been obtained had the calculations been based on variables expressed as deviations from their respective means. Augmented moments are moments that have been multiplied by the number of observations included in the analysis. They are used to avoid rounding errors.

Coefficient of determination.--The square of a correlation coefficient.

Column-by-column matrix multiplication.--A variant of the more common row-by-column computation (see p. 101).

Column vector.--A matrix consisting of m rows and one column (see p. 22).

Consistent estimates.--Estimates of statistical coefficients obtained in such a way that the average value for many large samples equals the value that would be obtained from a similar calculation based on the combined evidence of all possible samples. For unbiased estimates, the same property holds when estimates are made from samples of any size.

27/ Definitions given here refer only to terms included in this handbook. In cases where an exact definition would require a large amount of space, a condensed explanation is given instead, possibly with a cross-reference to another publication. The reader is presumed to be acquainted with terms covered in a first course in statistics that includes multiple and partial correlation and regression.

Covariance.--The joint sampling variation between two or more statistical coefficients. It is analogous to the variance of a single coefficient.

Cramer's rule.--A computational scheme for solving systems of equations. In this handbook, its use is recommended only for nonsymmetrical systems consisting of exactly 3 equations (see p. 96).

Critical values.--Points in a probability distribution that delineate a given percentage of the items. Points of 5 percent or 1 percent are commonly used in statistical tests.

Crout method.--A computational scheme for solving systems of equations, inverting matrices, and performing similar operations. In this handbook, it is used only when the matrix to be inverted is nonsymmetrical; but it can be used for any matrix (see p. 95).

Deadjusted data.--The original data on which the analysis was based, as contrasted with the adjusted moments used for computational purposes.

Degree of identification.--See identification.

Determinant.--A numerical value associated with a square matrix (see p. 26).

Diagonal of a matrix.--See main diagonal of a matrix.

Doolittle method.--A computational scheme for solving systems of equations, inverting matrices, and performing similar operations. As commonly used, it includes a forward and a back solution; in this handbook, in most instances only the forward part of the solution is used. As described in this handbook, the method can be applied only when the matrix to be inverted is symmetrical (see p. 89).

Efficient estimates.--Estimates of statistical coefficients obtained in such a way that their average standard error for many large samples is as small as possible. For "best" estimates, the same property holds when estimates are made from samples of any size.

Endogenous variables.--A set of variables that are assumed to be determined simultaneously by common economic forces. Lagged values of endogenous variables are included among predetermined variables in an analysis.

Exogenous variables.--A set of variables that are unaffected by the common economic forces that are assumed to affect endogenous variables in a system of equations.

Floating decimal point.--The carrying of a fixed number of significant figures in clerical computations, and therefore a varying number of decimal places, as contrasted with the method, suggested in this handbook, of carrying a fixed number of decimal places.

Full information approach.--A maximum likelihood method for deriving estimates of the structural coefficients for each equation in a system of equations. Estimates of all coefficients in all equations in the system are obtained simultaneously. In general, the computations involved are formidable so the method is seldom used.

Identification.--A mathematical property of an equation that indicates whether the structural coefficients can be estimated by statistical means. Degree of identification refers to whether the equation is underidentified, just identified, or overidentified.

Identity matrix.--A square matrix in which all elements on the main diagonal are 1, and all nondiagonal elements are zero (see p. 22).

Instrumental variable approach.--One of several methods for obtaining estimates of the structural coefficients that are statistically consistent in a single equation that contains more than one endogenous variable. A set of predetermined variables from the entire system is chosen such that one variable is available for each coefficient to be estimated. These are called instrumental variables. They are used in the manner described by Klein (13, pp. 122-125). The objection to this approach is that different answers are obtained depending on the particular set of instrumental variables used.

Inverse of a matrix.--The inverse of the matrix A is written as A^{-1}. The inverse of A is that matrix which when multiplied by A equals the identity matrix. If the original matrix is symmetrical, the inverse also is symmetrical (see p. 26).

Iterative method.--A computational device or formula in which an initial value is assumed or estimated and successive values are derived from the formula. Such methods are used only when the successive values are known to converge to the value that would be obtained by a direct computation.

Just identified equation.--An equation that has the mathematical property that permits a unique determination of its structural coefficients from regression coefficients in reduced form equations (see p. 83). Such equations can be fitted by the method of reduced forms or by a modification of the limited information approach.

Limited information approach.--A maximum likelihood method for deriving estimates of the structural coefficients for equations that are overidentified. The coefficients usually are estimated for one equation at a time, with the simultaneity implied by the system taken into account in the computations, but information on the particular variables that appear in each of the other equations in the system is ignored. The estimates are statistically consistent and as efficient as any others based on the same amount of information. A slight modification of this approach is used for equations that are just identified.

Main diagonal of a matrix.--The elements, listed in order, along the diagonal of a square matrix starting with a_{11} and ending with a_{nn}.

Matrix.--An array of numbers arranged in rows and columns (see p. 21). An (m x n) matrix consists of m rows and n columns.

Maximum likelihood.--A commonly used mathematical procedure for obtaining formulas to estimate statistical coefficients. Coefficients are derived in such a way as to maximize a likelihood function. The results are known to be statistically consistent and efficient. Statistical coefficients obtained from such formulas are called maximum likelihood estimates.

Model.--A system of related structural equations, together with some implied joint or combined probability distribution for their error terms.

Moment matrix.--As used in this handbook, this always means an augmented moment matrix.

Moments.--See augmented moments.

Monte Carlo approach.--A method used to indicate empirically the amount and kind of sampling variation that can be expected under given conditions. Samples are drawn from a known population, estimates based on these samples are made, and the results analyzed.

Order of a matrix or determinant.--The number of rows or columns in a square matrix or in a determinant.

Overidentified equation.--An equation that has the mathematical property that alternative estimates of its structural coefficients can be obtained from the regression coefficients in the reduced form equations (see p. 83). Hence the method of reduced forms cannot be used to estimate the coefficients; instead the limited information approach commonly is used.

Partially-reduced form equation.--In some equation systems data are unavailable for certain endogenous variables in the structural equations. In such cases, variables with which this variable is assumed to be related are substituted algebraically for it in other equations. The resulting equations are called "partially-reduced form equations" (see p. 74).

Predetermined variables.--A set of variables that are assumed to affect endogenous variables in a system of equations but not to be directly affected by them. They may include exogenous variables and lagged values of endogenous variables.

Rank of a matrix.--A matrix has rank r if the largest nonzero determinant included in the matrix is of order r.

Recursive approach.--Some systems of equations are formed such that at least one of them contains only a single endogenous variable. Consistent estimates of coefficients in such equations can be obtained by solving them directly by least squares. If other equations contain only one endogenous variable other than those contained in the first set of equations, consistent estimates of the coefficients in these equations can be obtained if they are solved directly by least squares, provided calculated values of the other endogenous variables are substituted for actual values before making computations. Systems of equations in which each equation can be fitted by least squares by the successive substitution of calculated values of endogenous variables are known as recursive systems, and this method of solving them is called the recursive approach. Although the estimates of the structural coefficients are statistically consistent, frequently they have larger standard errors than if estimated directly as in the limited information approach.

Reduced form equations.--Equations that result when each endogenous variable in a system of equations is written as a linear function of all of the predetermined variables in the system. In this handbook, they are used as a computational device when using the system of equations for analytical purposes (see p. 83).

Reduced form method.--A method that yields estimates of structural coefficients that are statistically consistent and efficient for equations that are just identified. Reduced form equations are solved by least squares and the structural coefficients obtained by an algebraic transformation. Results obtained are identical with those given by the modification of the limited information method described in this handbook.

Row-by-row matrix multiplication.--A variant of the more common row-by-column computation (see p. 101).

Row vector.--A matrix consisting of one row and n columns (see p. 22).

Scalar.--An ordinary number, as contrasted with a matrix or a vector. It can be thought of as a matrix with one row and one column.

Simultaneous correlation coefficient.--A coefficient that indicates the percentage of variation in the joint distribution of endogenous variables in a system of equations explained by all predetermined variables in the system (see p. 86).

Square matrix.--A matrix having the same number of rows and columns.

Structural equations.--Equations derived from the basic economic relationships that are assumed to prevail within a system of equations, as contrasted with other equations, such as reduced form equations, that are used chiefly as a computational device.

Symmetrical matrix.--A square matrix in which all the corresponding elements above the main diagonal are equal to elements below the diagonal; that is, $a_{ij} = a_{ji}$ (see p. 22).

Transpose of a matrix.--The transpose of the matrix A is written as A'. The transpose of A is a matrix in which rows of A are columns of A', and columns of A are rows of A'. The transpose of a symmetrical matrix equals the matrix itself (see p. 22).

Two-tailed test.--A statistical test involving both tails of a probability distribution. Critical values for a 2-tailed test at a 5-percent probability level are found such that $2\frac{1}{2}$ percent of the items lie to the left and right of these points respectively in each tail.

Underidentified equation.--An equation that has the mathematical property that its structural coefficients cannot be estimated by statistical means (see p. 29).

Unit matrix.--See identity matrix.

Variance.--The square of a standard error.

Vector.--A matrix consisting of only one row or one column. By convention, the term vector alone implies a row vector consisting of one row. A column vector consisting of one column is written as the transpose of a (row) vector (see p. 22).